Assortment Optimization Under History-Dependent Effects

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Abstract. This paper examines how to plan multi-period assortments when customer utility depends on historical assortments. We formulate this problem as a nonlinear integer programming model and show it is NP-hard in the presence of a negative history-dependent effect (such as a satiation effect). We build solution methodologies for obtaining global optimal solutions under a general setting that the history-dependent effects could be a mixture of positive and negative. We propose using a lifting-based framework to reformulate the problem as a mixed-integer exponential cone program that state-of-the-art solvers can solve. We also design a sequential revenue-ordered policy and show that it solves our problem to optimality in polynomial time when historical assortments positively affect customer utility (such as an addiction effect). Additionally, we identify an optimal cyclic policy for an asymptotic regime, and we also relate its length to the customer's memory length. Finally, we present a case study using a catering service dataset, showing that our model demonstrates good fitness and can effectively balance variety and revenue.

Key words: choice model, satiation, mixed-integer nonlinear programming, perspective formulation, convex extension, cyclic policy

1. Introduction

Many companies sell products to customers who make repeat purchases or visit over multiple periods. For example, corporate cafeterias provide workday luncheons for employees; online grocery flash sell platform plans assortments for customers who repeatedly visit the online store and seek deep discounts. Customers' utility may vary over time and may divert to products offered by competitors if they feel "bored". Therefore, these companies must carefully plan their daily assortments over time to keep their customers satisfied.

A typical example is the corporate dining industry. Corporate dining is an essential but often overlooked industry. It was reported that a working person will spend an average of $2500 per year on working-day lunches (ezCater 2024). Though corporate cafeterias are convenient and usually offer subsidized prices, they often face complaints of boredom due to the need for more dish variety. To address this, cafeteria managers manually diversify their menus based on their experience and industry knowledge. However, without a careful understanding of how previous menus impact customer demand, manual changes may be inefficient.

Many studies have explored and validated that customers' historical experiences may affect their willingness to buy or retain a product (Bowden 2009, Lemon and Verhoef 2016). For example, a customer may initially feel satisfied with a product, but this satisfaction may wane over time. This utility decay phenomenon is also observed in fashion products (Caro et al. 2014). On the other hand, a customer may also prefer to consistently consume "traditional and familiar" food that could maintain or increase her utility (Edenred

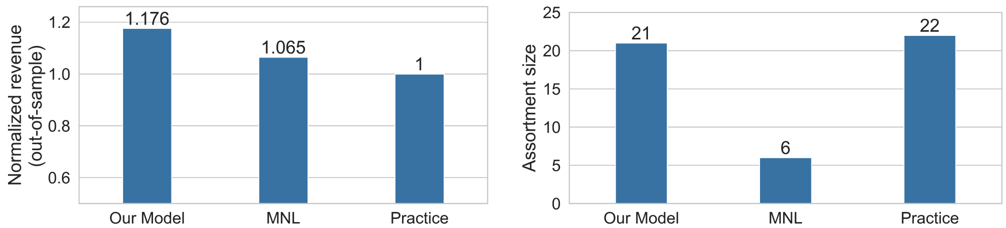
2024). Today, with the increasing collection of transactional data, companies seek to leverage these data to estimate customers' utility changes and build data-driven assortment planning systems. For instance, new technology companies have emerged that revolutionize traditional catering services by analyzing transactional data to estimate customers' preferences and offering optimized daily cafeteria menus (Fooda 2024).

Besides corporate cafeterias, analogous situations arise in various other sectors. Flash sales platforms must organize their promotional assortments ahead of time, based on the products suppliers provide (Martínez-de Albéniz et al. 2020). For instance, Meituan, a leading Chinese online shopping platform, manages a grocery flash sale channel. Due to the lead time of contract negotiations with suppliers, Meituan needs to decide, in advance, assortments for daily deals over a fixed planning horizon. Given that customers frequently visit the channel for discounted groceries, planning must consider historical offerings. More details about Meituan's flash sale campaign are in the ecompanion EC.1. Managers of cafeterias and flash sales encounter a common multi-period planning problem that decides how to offer identical assortments to all customers over a fixed planning horizon. In practice, these assortments are usually predetermined before the planning horizon, possibly due to limits (such as lead time) from the supplier side.

In this paper, we focus on the above applications and investigate the corresponding *multi-period assortment planning problem considering history-dependent utilities* to maximize the long-run average revenue. The history-dependent effects bring new challenges for assortment planning because of the interplay between cross-product and cross-period effects. Cross-product effects refer to the influence of other products in an assortment on purchase decisions. For instance, customer substitution behavior is commonly modeled using discrete choice models, like the multinomial logit (MNL) model (McFadden 1974). Cross-period effects refer to the temporal influences on customer utility, i.e., the utility of a product in a given period depending on its offering history. Given both types of effects, the offering decisions for multiple products over multiple periods are made simultaneously. The number of candidate assortments, therefore, scales exponentially with the number of products and the planning horizon. This can be enormous for even a practically-sized problem. Moreover, customer behavior under both types of effect makes the decision more complex because, for a given period, the purchase probability is a nonlinear function—specifically, a softmax function—of the history-dependent utility functions.

Despite the complexity of assortment planning with history-dependent effects, the benefit of considering such effects is significant. Using the data from a corporate cafeteria with which we cooperated, we evaluate assortments derived from a history-dependent model (our model, estimated in Section 5), from the MNL model, and the current practice observed in the data (as unit one for normalization). The results shown in Figure 1 indicate that our model achieves the highest revenue in the out-of-sample test, about 10.4% more than the revenue generated by the MNL model. Our model also produces assortments that have greater variety than those obtained by the MNL model. The current practice decided by managers resulted in assortments with a greater variety but the lowest revenue. See Section 5 for details of the case study.

Figure 1 The average out-of-sample results of three assortment plans



Note. Our model is the history-dependent model. We normalize the revenue generated by the current practice to one. The assortment size is the number of product types offered in the planning horizon.

1.1. Description of Results

We study a multi-period assortment planning problem in which a firm facing both cross-product and cross-period effects needs to decide what to offer in each period to maximize the average expected revenue. We model customer choice in a given period using a variant of the MNL model in which the utility of each product linearly depends on its past offering decisions. In our model, an assortment decision for a given period affects both current and future purchase behavior in a nonlinear fashion. In other words, our assortment planning problem is a nonlinear integer programming problem.

We show that this problem is NP-hard under the presence of negative history-dependent effects and propose exact solution methodologies for a general setting that the history-dependent effects could be a mixture of addiction and satiation. We propose a lifting-based reformulation framework to obtain a mixed-integer exponential cone program (MIECP) formulation. The key idea behind our formulation is to lift the original nonconvex constraints into a higher-dimensional space where the lifted constraints are simpler to model. This reformulation allows us to utilize prevalent techniques in the optimization literature, such as the McCormick envelope, the perspective formulation, and the Lovász extension, to model the lifted structures. Numerical studies indicate that our formulation efficiently achieves optimality and yields higher revenue than heuristic policies. We also identify an optimal sequential revenue-ordered (RO) policy if history-effects are all positive, i.e., addiction effects. This characterization indicates that our problem with the addiction effect is solvable in polynomial time. For problems with a long-term planning horizon, theoretically, we show that a cyclic policy is optimal and find that, under a certain non-overlapping condition, the optimal cycle length equals the customer memory length plus one. Computationally, finding such policies is NP-hard but also admits an MIECP formulation. Moreover, the non-overlapping condition leads to a mixed-integer linear programming (MILP) formulation that is stronger than the conic one.

We conduct a case study using transactional data from a corporation cafeteria operated by a high-tech catering service company in China. The cafeteria primarily serves workday lunches for employees who work nearby. We estimate customers' utility function and then design weekly day-by-day lunch menus. The estimation shows that a satiation effect does exist, but it only lasts for a short period (i.e., two or three days).

We demonstrate that our model can provide menus that balance variety and revenue, while current industry practices sacrifice revenue, and the classic MNL model yields menus with limited variety and low revenue.

We highlight the contributions of our paper as follows:

1. Our model, which we refer to as HAP (i.e., history-dependent assortment problem), simultaneously incorporates the cross-product and cross-period effects into the multi-period assortment planning process. The cross-period history-dependent effects could be all positive, all negative, and mixed across products and periods. We demonstrate, via a case study of a corporate cafeteria, that our model has several benefits. First, our result suggests that the service provider should provide a greater variety of products if satiation effects exist. When satiation effects last longer, it is beneficial to offer more types of products across periods. Furthermore, it is recommended that a product be offered only once within the memory length to prevent a utility decrease due to satiation effects, particularly in the case of products with high prices. This observation naturally leads to a "non-overlapping" cyclic assortment planning policy, which we explore in Section 4, where we establish its optimality. Most importantly, our history-dependent policy effectively balances variety and the revenue objective, resulting in assortments that achieve the highest revenue in out-of-sample tests while maintaining the same variety as found in practical scenarios.

2. We show that finding a global optimal solution is NP-hard. To tackle this difficulty, we develop an MIECP formulation of our HAP model by using lifting and coupling ideas from the optimization literature, including perspective formulation, the McCormick envelope, and the Lovász extension. In particular, we modify the Lovász extension to model attraction values under general history-dependent effects. Our formulations, together with cutting-plane algorithms, outperform an MILP formulation of HAP obtained using a prevalent strategy in the mixed-integer nonlinear programming (MINLP) literature and the original nonlinear integer formulation HAP solved by the state-of-the-art MINLP solver `SCIP`. Additionally, we characterize, in terms of history-dependent effects, the conditions under which a sequential revenue-ordered policy globally solves our HAP model.

3. We establish the asymptotic optimality of cyclic policies, and finding such policies allows an MIECP formulation. Our study reveals a connection between the length of an optimal cyclic policy and the length of customer memory. In particular, we characterize the condition under which the optimal cyclic length is the customer memory length plus one. This condition, in addition, leads to an MILP formulation for finding optimal cyclic policies, which is, theoretically, tighter than the previous conic formulation.

1.2. Literature Review

Extensive research has incorporated history-dependent effects into operations problems, including pricing and promotion (Bi et al. 2025, Chen et al. 2023c), service design (Bernstein et al. 2022, Lei et al. 2023), resource allocation (Adelman and Mersereau 2013), retention management (Kanoria et al. 2024), etc. Studies show that ignoring history-dependent effects may lead to substantial losses. For instance, considering the reference price can reduce 5.82% prediction error of sales (Wang 2018), and the optimal bundle accounting for customer satiation improves the revenue by more than 4.5% (Gürlek et al. 2025).

History-dependent effects are receiving increasing attention in assortment planning. One stream of work is to design personalized assortment planning for online platforms. The history-dependent effect derives from repeated interactions between the firm and customers, which breaks the traditional single-interaction assumption in personalized assortment optimization literature (El Housni and Topaloglu 2023). Based on the easily accessible purchase history of a customer, these studies consider two settings: (i) customers have stable repeated interactions with the firm, e.g., Chen et al. (2023b) optimize adaptive assortment decisions based on information inferred from a customer's historical purchases; (ii) a customer can leave or churn if he or she is not satisfied. Hence, many studies focus on designing delicate exploration-exploitation recommendation algorithms that may limit exploration budgets (Bastani et al. 2022, Sumida and Zhou 2023).

For non-personalized assortment planning, the history-dependent effect is captured by a dynamic choice/demand function of the entire market. The effect may derive from attributes of a product itself, e.g., declining utility. Such intertemporal connections lead to a nonlinear and high-dimensional problem. Two settings are considered: (i) an adaptive assortment policy, e.g., Caro and Martínez-de Albéniz (2020) consider a content release problem where the attraction of content decays and the number of followers stochastically evolves; (ii) a non-adaptive assortment plan for the entire horizon made before the first period, e.g., Caro et al. (2014) optimize release schedule of fast fashion products with exponentially declining attraction before period one due to a long lead time. Our work contributes to the second type of literature. Our model can represent the model in Caro et al. (2014) and adds more flexibility (see details in Section 2.2), which brings new challenges and needs additional methods.

Our paper contributes to the growing literature on developing new solution methods for assortment planning. The three primary approaches include the revenue-ordered policy, approximation algorithms, and integer programming for obtaining exact solutions. Since Talluri and Van Ryzin (2004) showed that the optimal assortment under the MNL model is revenue-ordered, this policy has been extensively studied in various contexts (Wang and Wang 2017, Sumida and Zhou 2023). For example, Gao et al. (2021) prove that the revenue-ordered assortment policy is optimal under the MNL model when the offered assortment is gradually revealed. Xu and Wang (2023) exploit the revenue-ordered property to solve a multistage assortment optimization problem. We contribute to this literature by showing that a sequential revenue-ordered policy is optimal for our problem under the addiction effect.

A stream of studies focuses on designing approximation algorithms for assortment plannings. For instance, Feldman and Topaloglu (2015) and Désir et al. (2020) design constant factor approximation algorithms for capacitated assortment optimization under the nested logit model and the Markov chain choice model, respectively. Later, Désir et al. (2022) propose a fully polynomial time approximation scheme (FPTAS) for capacitated assortment problems under the mixed MNL, Markov chain, and nested logit choice models. FPTAS are also proposed to solve assortment optimization under the exponential choice model (Aouad et al. 2023), with online dynamic behavior (Liu et al. 2020, Feldman and Jiang 2023, Aouad et al. 2024), and with multi-purchase behavior (Jasin et al. 2024, Luan et al. 2025).

Our approach is mostly related to the third direction — using integer programming techniques to obtain global optimal assortment decisions. A stream of studies establishes MILP models for assortment planning under various choices models, including the ranking-based choice model (Bertsimas and Mišić 2019), the mixture of Mallows model (Désir et al. 2021), the decision forest choice model (Akchen and Mišić 2025), and the mixture-of-nested-logit model (Fan et al. 2024). In particular, Chen et al. (2025) conduct polyhedral studies for the MNL-based assortment planning in the context of quick-commerce. Mixed-integer conic programming has also been used to model choice behavior. Sen et al. (2018) propose a mixed-integer second-order cone programming formulation for assortment problems under the mixed MNL model, and Chen et al. (2022) use this idea to solve location-dependent offline-channel assortment planning in omnichannel retailing. Akçakuş and Mišić (2025) develop various mixed-integer exponential cone programs to solve a share-of-choice product design problem where customers follow a logit-based choice model. Our solution approach is inspired by formulations in Akçakuş and Mišić (2025). However, due to the interaction of cross-product and cross-period effects, additional ideas are required for our problem, including such relaxation techniques as McCormick envelope (McCormick 1976) and the Lovász extension (Lovász 1983).

Last, we comment that our paper contributes to a growing set of applications of conic programming/discrete optimization in designing operations systems, including joint inventory-location problems (Atamtürk et al. 2012), battery swap networks design (Mak et al. 2013), service delivery scheduling (Kong et al. 2013), process flexibility design (Simchi-Levi and Wei 2012, Yan et al. 2018), electric vehicle charging (Chen et al. 2023a), proactive policing (He et al. 2025), network flow (Simchi-Levi et al. 2019), and nonlinear resource allocation (He and Tawarmalani 2024).

1.3. Structure

The rest of the paper is organized as follows. In Section 2, we formally define the assortment planning problem and establish its computational complexity. In Section 3, we propose a mixed-integer exponential cone formulation that can handle any mixed addiction-satiation history-dependent effects. We prove that, under positive effects, a revenue-ordered policy is optimal. In Section 4, we present our results on cyclic policies. We present a case study using real data in Section 5 and report numerical results on synthetic data in Section 6. We conclude the paper in Section 7. All proofs are given in the ecompanion.

2. Model

In Section 2.1, we define our multi-period assortment planning problem with history-dependent effects and establish its computational complexity. In Section 2.2, we discuss the model features in detail.

2.1. Incoporating History-Dependent Effects into Choice Models

Consider a firm that sells N products to customers. Let $[N] := \{1,\ldots,N\}$ denote the set of all products, and let $[N]^+ := [N] \cup \{0\}$, where the index 0 denotes the outside option. For each product $i \in [N]$, let r_i denote the revenue obtained from selling an instance of product i. There is a total of T planning horizons, and the firm needs to decide what to offer in each period. More specifically, for each period $t \in [T]$, let $\boldsymbol{x}^t = (x_1^t,\ldots,x_N^t) \in \{0,1\}^N$ be a binary decision variable modeling the subset of products offered in period t, i.e., $x_i^t = 1$ if and only if product i is offered in period t. We assume that in each period, one unit mass of customers arrives, and the customers select an item from the offered products or leave without a purchase. For example, in a setting such as corporate cafeterias, the firm serves relatively stable customers (e.g., most customers are employees who work nearby), and customers make purchase decisions on a daily basis.

We modify the MNL model to encapsulate the impact of the historical assortments on customer choices. Each product has a utility U_i, which depends on attributes of the product and environment. Following Guadagni and Little (1983), Wang (2018) and Xie and Wang (2024), we assume that the utility of one product is a linear function of its attributes and historical offerings. Specifically, the utility of product i in period t affinely depends on its historical offering records within a memory length M:

$$U_i^t = \beta_i^0 + \sum_{m\in[M]} x_i^{t-m}\beta_i^m, \tag{Utility}$$

where $M \in \mathbb{Z}_+$ denotes the customer memory length. For simplicity in the presentation, we assume throughout this paper that $x_i^k := 0$ for $k \le 0$. The parameter β_i^0 is the base utility of the product i determined by its attributes. The parameter β_i^m is the incremental utility if product i is offered in the m^{th} period before the current time. Let $\boldsymbol{\beta} := (\beta_i^m)_{i\in[N],m\in[M]}$ denote the history-dependent effect matrix. The value of β_i^m can be positive or negative. A positive β_i^m indicates that exposure to product i increases its utility, acting as a stimulus (Fox et al. 1997). While a negative β_i^m suggests a satiation effect, leading to customer fatigue from repeated offerings. The history-dependent effects can vary across products and periods. In our planning model, the assortments are fixed for each period. Therefore, we do not consider personalized assortments or model individual-level utility functions.

For a given assortment plan $\boldsymbol{x} := (\boldsymbol{x}^1,\cdots,\boldsymbol{x}^T)$, we use $\pi_i^t(\boldsymbol{x}^t,\ldots,\boldsymbol{x}^{t-M})$ to denote the purchase probability of product i in period t. Under the MNL model, the purchase probability of product i in period t is

$$\pi_i^t(\boldsymbol{x}^t,\ldots,\boldsymbol{x}^{t-M}) = \frac{x_i^t \exp\left(\beta_i^0 + \sum_{m\in[M]} x_i^{t-m}\beta_i^m\right)}{1+\sum_{j\in[N]} x_j^t \exp\left(\beta_j^0 + \sum_{m\in[M]} x_j^{t-m}\beta_j^m\right)} \quad \text{for } i \in [N],$$

$$\pi_0^t(\boldsymbol{x}^t,\ldots,\boldsymbol{x}^{t-M}) = \frac{1}{1+\sum_{j\in[N]} x_j^t \exp\left(\beta_j^0 + \sum_{m\in[M]} x_j^{t-m}\beta_j^m\right)},$$

where we normalize the utility of the outside option to 0. The purchase probabilities depend on two components: the current assortment $\boldsymbol{x}^t$ and the past assortments $\boldsymbol{x}^{t-1},\ldots,\boldsymbol{x}^{t-M}$. The former dependence captures cross-product effects such as the substitution effect, while the latter dependence models cross-period effects. The purchase probability can also be viewed as the market share of product i in period t and presents the effect of historical assortments on sales of the entire market (Batsell and Polking 1985).

With these definitions, assortment planning under history-dependent effects can be expressed as the following fractional binary program, called the history-dependent assortment problem:

$$\max\left\{\sum_{t\in[T]}\frac{1}{T}\sum_{i\in[N]} r_i\pi_i^t(\boldsymbol{x}^t,\ldots,\boldsymbol{x}^{t-M})\,\middle|\,\boldsymbol{x}^t\in\mathcal{X}\cap\{0,1\}^N \text{ for } t\in[T] \text{ and } \boldsymbol{x}\in\mathcal{P}\right\}, \tag{HAP}$$

where $\mathcal{X}$ (resp. $\mathcal{P}$) is a system of linear inequalities that models cross-product constraints (resp. cross-period constraints). The cross-product constraints often include resource restrictions within a single period, such as cardinality and capacity constraints, and address various operational considerations in the assortment planning literature (Sen et al. 2018, Chen et al. 2025). Cross-period constraints describe the intertemporal relationships of assortments across different periods, such as non-overlapping assortments over consecutive periods, which occur in multi-period assortment planning problems (Liu et al. 2020, Chen et al. 2023b).

This model is generally difficult to solve. We show in the following proposition that (HAP) is NP-hard under very simple settings if history-dependent effects are negative.

PROPOSITION 1. (HAP) *is NP-hard even when the planning horizon is two, the memory length is one, and history-dependent effects are negative.*

2.2. Model Discussion

We outline here the main features of our model, justify our assumptions, and identify relevant scenarios.

It is apparent that our formulation requires a firm to present the same assortment to all customers in each period before the planning horizon begins. In other words, the firm is unable to take advantage of personalized assortments and adaptive policies, which are common practices in online assortment optimization such as Amazon's recommendation system. However, those practices are not suitable for the case of corporate cafeterias, where it is common to treat employees with the same menu, and it is impractical to dynamically adjust assortments due to the lead time in both procurement and cooking processes (Van Wezel et al. 2006). Observations from our partnered catering company indicate that the corporate cafeteria plans the lunch menus on a weekly basis. Moreover, our model assumption is also reasonable for flash sale channels (i.e., predetermine assortments for several days due to the negotiation time with suppliers).

Our model describes the history-dependent effect through a linear utility function with historical offerings as features. The linear utility function assumption is common in the logit-based choice model to capture impacts such as product attributes (Guadagni and Little 1983) and historical prices (Wang 2018). Moreover, the linear utility function covers multiplication in the attraction value in literature. For example, our model with negative history-dependent effects represents the model in Caro et al. (2014). Our model offers additional flexibility by allowing for positive, negative, and mixed history-dependent effects and offering products in any period.

In the linear utility function, we incorporate the historical offerings to capture the history-dependent effect. The utility function captures the direct effect of historical assortments that are critical decisions at a firm level. Moreover, we only consider the direct history-dependent effect, i.e., the historical offering of a product will only affect the product's future utility itself. We do not include the cross-product history-dependent effect, i.e., the historical offering of a product might affect the utility of other products, because this effect is usually at a much smaller scale (e.g., Vilcassim et al. 1999, Dubé and Manchanda 2005, Mehta 2007). In our empirical estimation, the cross-product history-dependent effect is approximately one-tenth of the direct effect and is significant only when $M = 1$. We believe that our model efficiently captures the primary effect and maintains model tractability.

We manage the history-dependent effects across periods through the memory length M, inspired by the patience level from dynamic pricing studies (Popescu and Wu 2007, Liu and Cooper 2015). This patience level indicates how many periods a customer will wait for a price reduction. Consequently, it is logical to assume that customers might retain a limited memory length for a product.

Our model does not include inventory decisions. In cafeteria settings, for instance, procurement choices can be made subsequent to assortment planning. Additionally, inventory rollover is not considered in the study because cooked dishes cannot be kept for the following day due to hygiene standards. To simplify the analysis, this study will concentrate on the assortment planning problem with cross-product and cross-period effects, which is already a challenging problem.

3. Solution Approaches

In this section, we aim to solve the history-dependent assortment problem (HAP) to its global optimality. In Section 3.1, we propose a mixed-integer exponential cone formulation for the general case, and, in Section 3.2, we characterize conditions under which a sequential revenue-ordered policy is optimal.

3.1. Integer Programming Approach

Due to the combinatorial and nonconvex nature of (HAP), state-of-the-art global optimization solvers such as `BARON` (Tawarmalani and Sahinidis 2005) and `SCIP` (Bestuzheva et al. 2021) fail to solve practical-sized instances—see Section 6.1 for computational evidence. Nevertheless, we focus on developing techniques for obtaining mixed-integer programming formulations of (HAP), which will allow us to leverage modern mixed-integer program solvers such as `Gurobi` and `MOSEK`. Then, in Section 4, we will use these techniques to find asymptotically optimal cyclic policies.

Our formulation introduces the following additional variables to represent the customer choice behavior. For each period $t \in [T]$, we use ρ^t to denote the no-purchase probability for period t:

$$\rho^t = \frac{1}{1 + \sum_{j \in [N]} x_j^t \exp\left(\beta_j^0 + \sum_{m \in [M]} x_j^{t-m} \beta_j^m\right)},$$

and for each product $i \in [N]$, we use y_i^t to denote the choice probability for product i in period t:

$$y_i^t = \frac{x_i^t \exp\left(\beta_i^0 + \sum_{m\in[M]} x_i^{t-m} \beta_i^m\right)}{1 + \sum_{j\in[N]} x_j^t \exp\left(\beta_j^0 + \sum_{m\in[M]} x_j^{t-m} \beta_j^m\right)}.$$

In what follows, we refer to ρ^t as the no-purchase probability variable and y_i^t as the choice probability variable. Using these definitions, we can reformulate (HAP) as follows:

$$\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{\rho}} \quad & \frac{1}{T} \sum_{i\in[N]} \sum_{t\in[T]} r_i y_i^t \\
\text{s.t.} \quad & \boldsymbol{x}^t \in \{0,1\}^N \cap \mathcal{X} \text{ and } \boldsymbol{x} \in \mathcal{P} && \text{for } t \in [T] \\
& \rho^t + \sum_{i\in[N]} y_i^t = 1 && \text{for } t \in [T] \\
& y_i^t = \rho^t x_i^t \exp\Big(\beta_i^0 + \sum_{m\in[M]} x_i^{t-m} \beta_i^m\Big) && \text{for } t \in [T] \text{ and } i \in [N]. \qquad \text{(CHOICE)}
\end{aligned}$$

The objective function is a non-negative weighted combination of choice probability variables. The second constraint ensures that the sum of the choice probability and the no-purchase probability equals one for each period. The last constraint represents the choice probability for product i in period t, and thus we refer to it as constraint (CHOICE). This reformulation encapsulates the nonconvexity of problem (HAP) in constraint (CHOICE). Hence, it suffices to derive a convex representation of constraint (CHOICE).

The main component of constraint (CHOICE) is the composition of a univariate exponential function and an affine utility function, defined as follows:

$$\alpha_i(z^1, \ldots, z^M) := \exp\Big(\beta_i^0 + \sum_{m\in[M]} \beta_i^m \cdot z^m\Big) \qquad \text{for } (z^1, \ldots, z^M) \in \{0,1\}^M.$$

For a given assortment plan $\boldsymbol{x} := (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^T)$, we interpret $\alpha_i(x_i^{t-1}, \ldots, x_i^{t-M})$ as the attractiveness of product i in period t. Therefore, we will henceforth refer to $\alpha_i(\cdot)$ as the attraction value function. To handle the discrete nature of the attraction value function, we will use the notion of a *continuous extension*. Given a function $f : \{0,1\}^n \to \mathbb{R}$, a function $g : [0,1]^n \to \mathbb{R}$ is called a continuous extension of $f(\cdot)$ if $g(\cdot)$ is continuous and $g(\boldsymbol{z}) = f(\boldsymbol{z})$ for every $\boldsymbol{z} \in \{0,1\}^n$. In addition, if an extension function $g(\cdot)$ is convex (resp. concave), we say that it is a convex (resp. concave) extension. It is common for a discrete function to admit many convex (resp. concave) extensions. A natural idea is to use the tightest convex/concave extension. Since the discrete domain $\{0,1\}^n$ is the vertex set of the continuous domain $[0,1]^n$, the pointwise largest convex (resp. smallest concave) extension of $f(\cdot)$ is the *convex (resp. concave) envelope*, denoted as $\operatorname{conv}(f)(\cdot)$ (resp. $\operatorname{conc}(f)(\cdot)$), of $f(\cdot)$ over the box $[0,1]^n$ (see Tawarmalani and Sahinidis 2002, Theorem 6).

3.1.1. A Lifted Representation of (CHOICE) In this subsection, we focus on modeling constraint (CHOICE). The idea is to lift the original nonconvex constraint (CHOICE) into a higher-dimensional space where the lifted constraint is dramatically simpler to represent than the original constraint. Specifically, this will allow us to invoke prevalent techniques, such as the McCormick envelope (McCormick 1976), the perspective formulation (Günlük and Linderoth 2010), and the Lovász extension (Lovász 1983), to model the lifted nonconvex structures.

Lifting for Simplicity The challenge in lifting is to identify a proper lifting space. To lift, we adapt a construction that Akçakuş and Mišić (2025) use to solve the logit-based product design problem, which we briefly review here. Consider a nonconvex constraint given as $\lambda + \lambda f(\boldsymbol{x}) \leq 1$, where $\lambda \geq 0, \boldsymbol{x} \in \mathbb{R}^n$, and $f(\cdot)$ is a convex function. This constraint can be relaxed into a convex constraint as follows. First, we multiply and divide $\boldsymbol{x}$ by λ to obtain

$$\lambda + \lambda f\left(\frac{\lambda \cdot \boldsymbol{x}}{\lambda}\right) \leq 1,$$

where $\lambda f(\frac{\lambda \cdot \boldsymbol{x}}{\lambda}) = 0$ when $\lambda = 0$. Next, we replace $\lambda \boldsymbol{x}$ with a new variable vector $\boldsymbol{u}$. This leads to

$$\boldsymbol{u} = \lambda \boldsymbol{x} \quad \text{and} \quad \lambda + \lambda f\left(\frac{\boldsymbol{u}}{\lambda}\right) \leq 1,$$

where the first constraint can be linearized using the McCormick envelope (McCormick 1976) and the second constraint is convex, since the function $\lambda f(\boldsymbol{u}/\lambda)$ is the *perspective function* of the convex function $f(\cdot)$ (Rockafellar 1970). Formally, given a function $f: D \to \mathbb{R}$, where D is a bounded subset of $\mathbb{R}^n$, the *perspective function* of f, denoted as $\operatorname{pers}(f)$, is defined as

$$\operatorname{pers}(f)(\lambda, \boldsymbol{x}) = \begin{cases} \lambda f(\frac{\boldsymbol{x}}{\lambda}) & \text{for } \lambda > 0 \text{ and } \boldsymbol{x} \in \lambda \cdot D \\ 0 & \text{for } \lambda = 0 \text{ and } \boldsymbol{x} = \boldsymbol{0}. \end{cases}$$

Perspective reformulations have been widely used in solving nonconvex optimization problems (Günlük and Linderoth 2010); see Bestuzheva et al. (2023) for a review of applications and a detailed computational study of perspective reformulations.

We now present our lift of the constraint (CHOICE). Note that the choice probability of product i in period t is computed by scaling its attraction value function $\alpha_i(\cdot)$ with a nonlinear factor $\rho^t x_i^t$, that is,

$$y_i^t = (\rho^t x_i^t) \cdot \alpha_i(x_i^{t-1}, \dots, x_i^{t-M}).$$

First, we introduce a new variable γ_i^t to represent the nonlinear factor $\rho^t x_i^t$. Second, we use the new variable γ_i^t to scale the historical assortment decisions x_i^{t-m} with $m \in [M]$, and we introduce the variable z_{im}^t to represent the scaled variable $\gamma_i^t x_i^{t-m}$. Using the additional variables γ_i^t and $\boldsymbol{z}_i^t = (z_{i1}^t, \dots, z_{iM}^t)$ and the definition of a perspective function, we decompose constraint (CHOICE) into constraints

$$\gamma_i^t = \rho^t x_i^t \tag{2a}$$

$$z_{im}^t = \gamma_i^t x_i^{t-m} \quad \text{for } m \in [M] \tag{2b}$$

$$y_i^t = \operatorname{pers}(\alpha_i)(\gamma_i^t, \boldsymbol{z}_i^t). \tag{2c}$$

Constraint (2a) lifts the nonlinear scaling factor $\rho^t x_i^t$ to γ_i^t; constraint (2b) lifts the scaled historical assortment decision $\gamma_i^t x_i^{t-m}$ to z_{im}^t; and constraint (2c) perspectifies the attraction value function in the space of the newly introduced variables γ_i^t and $\boldsymbol{z}_i^t$.

To obtain a mixed-integer convex formulation of constraint (CHOICE) in the lifted space, we individually relax each of the three constraints in (2). Specifically, we use the McCormick envelope (McCormick 1976) to relax constraint (2a). To apply the McCormick envelope to our setting, we derive a lower bound ρ_L^t and upper bound ρ_U^t on the no-purchase probability variable ρ^t, and impose the following constraints:

$$\begin{aligned} \gamma_i^t &\leq \min\big\{\rho_L^t x_i^t + \rho^t - \rho_L^t,\ \rho_U^t x_i^t\big\} \\ \gamma_i^t &\geq \max\big\{\rho_L^t x_i^t,\ \rho_U^t x_i^t + \rho^t - \rho_U^t\big\}. \end{aligned} \tag{3}$$

Similarly, using 0 (resp. ρ_U^t) as a lower (resp. upper) bound on γ_i^t, the McCormick envelope reformulates constraint (2b) as follows:

$$\begin{aligned} z_{im}^t &\leq \min\big\{\gamma_i^t, \rho_U^t x_i^{t-m}\big\} && \text{for } m \in [M] \\ z_{im}^t &\geq \max\big\{0,\ \rho_U^t x_i^{t-m} + \gamma_i^t - \rho_U^t\big\} && \text{for } m \in [M]. \end{aligned} \tag{4}$$

Finally, we represent constraint (2c) by using two continuous extensions of the attraction value function $\alpha_i(\cdot)$. On one side, a natural continuous extension $\tilde{\alpha}_i : [0,1]^M \to \mathbb{R}$ can be obtained from $\alpha_i(\cdot)$ by relaxing the binary assortment variables to continuous ones. This continuous function is convex since it is the composition of a convex function and an affine function. Using this extension, we relax constraint (2c) to

$$y_i^t \geq \operatorname{pers}(\tilde{\alpha}_i)(\gamma_i^t, \boldsymbol{z}_i^t).$$

This is a convex inequality since $\tilde{\alpha}_i(\cdot)$ is a convex function and so is its perspective function (Rockafellar 1970). Moreover, this constraint can be represented as an exponential cone constraint (Ben-Tal and Nemirovski 2001) (see more details in the ecompanion EC.3.2):

$$y_i^t \geq \exp(\beta_i^0) \cdot \gamma_i^t \cdot \exp\left(\frac{\sum_{m \in [M]} \beta_i^m z_{im}^t}{\gamma_i^t}\right), \tag{5}$$

where the right-hand-side function can be regarded as a convex under-estimator of the choice probability variable y_i^t. On the other side, an exact representation of (2c) requires us to overestimate the choice probability variable. To do this, we replace $\alpha_i(\cdot)$ with its tightest concave extension or, equivalently, concave envelope. Using the concave envelope, we arrive at another convex relaxation of (2c):

$$y_i^t \leq \operatorname{pers}\big(\operatorname{conc}(\alpha_i)\big)(\gamma_i^t, \boldsymbol{z}_i^t). \tag{6}$$

Next, we show that an explicit linear description of this constraint can be obtained from perspectifying a variant of the Lovász extension of $\alpha_i(\cdot)$ (Lovász 1983).

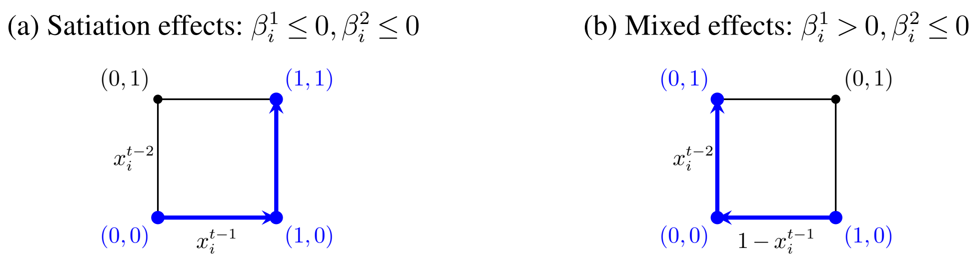
The Concave Envelope of $\alpha_i(\cdot)$ Our construction is based on the Lovász extension of a set function. The Lovász extension is an extension of a function f defined on $\{0,1\}^n$ to a function $\hat{f}$ defined on $[0,1]^n$. For each subset S of $[n]$, let $\chi^S \in \{0,1\}^n$ be its indicator vector—that is, the i^{th} coordinate of χ^S is 1 if and only if $i \in S$. Observe that every vector $\boldsymbol{x} \in [0,1]^n$ can be expressed uniquely as $\boldsymbol{x} = \lambda_0\chi^{T_0} + \lambda_1\chi^{T_1} + \cdots + \lambda_n\chi^{T_n}$, where $\lambda_k \geq 0$ for $k = 0,1,\ldots,n$ and $\sum_{k=0}^n \lambda_k = 1$, and $\emptyset = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n = [n]$. Thus,

$$\hat{f}(\boldsymbol{x}) = \lambda_0 f(\chi^{T_0}) + \lambda_1 f(\chi^{T_1}) + \cdots + \lambda_n f(\chi^{T_n})$$

is a well-defined extension of the function f (called the Lovász extension of f) on the continuous domain $[0,1]^n$. The Lovász extension serves as a bridge between supermodularity and concavity: f is supermodular if and only if its Lovász extension $\hat{f}$ is concave (Lovász 1983). Moreover, when the Lovász extension $\hat{f}$ is concave, it is expressible as the pointwise minimum of affine functions (Tawarmalani et al. 2013).

When the history-dependent effects are negative, the attraction value function $\alpha_i(\cdot)$ is supermodular, and its concave envelope is given by its Lovász extension (Tawarmalani et al. 2013). The key idea behind the closed-form description of the Lovász extension is to interpolate the attraction value function affinely over the nested historical assortments $\chi^{T_0}, \chi^{T_1}, \ldots, \chi^{T_M}$, where $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_M$. These nested assortments can be thought of as a path through the set of all possible past assortments $\{0,1\}^M$ going from $(0,0,\ldots,0)$ to $(1,1,\ldots,1)$, with each step along one coordinate direction. For example, when the memory length is two, the nested assortments $(0,0)$, $(1,0)$, and $(1,1)$ correspond to the blue path in Figure 2a. This path starts at $(0,0)$, moves along the first coordinate, then the second, and finally reaches $(1,1)$. To treat the general case of mixed effects, we modify the nested assortments—or equivalently the path—as follows: Instead of moving from $(0,0,\ldots,0)$ to $(1,1,\ldots,1)$, we start at χ^I and end at $\chi^{M\setminus I}$, where I denotes the indices of positive effects. For instance, consider mixed effects with $\beta_i^1 > 0$ and $\beta_i^2 \leq 0$. The blue path in Figure 2b, starting from $(1,0)$ and ending at $(0,1)$, corresponds to the past assortments that we will use in this case.

Figure 2 Illustration of switched nested assortments under two different history-dependent effects



Now, we formally define the historical assortments used in our construction. For each product $i \in [N]$, let I_i denote the indices of positive effects, that is, $I_i := \{m \mid \beta_i^m > 0\}$. Let Ω denote all permutations of past periods $[M]$. For each product $i \in [N]$ and a permutation $\sigma \in \Omega$, we define $M+1$ historical assortments, denoted as $\boldsymbol{h}_{i,0}^\sigma, \boldsymbol{h}_{i,1}^\sigma, \ldots, \boldsymbol{h}_{i,M}^\sigma$, as follows: $\boldsymbol{h}_{i,0}^\sigma = \chi^{I_i}$ and for each $k \in [M]$

$$\boldsymbol{h}_{i,k}^\sigma = \begin{cases} \boldsymbol{h}_{i,k-1}^\sigma + \boldsymbol{e}_{\sigma(k)} & \text{if } \sigma(k) \notin I_i \\ \boldsymbol{h}_{i,k-1}^\sigma - \boldsymbol{e}_{\sigma(k)} & \text{if } \sigma(k) \in I_i, \end{cases} \tag{SwitchNested}$$

With this definition, we are ready to describe the concave envelope of $\alpha_i(\cdot)$ which, after perspectification, yields an explicit linear description for the constraint (6).

PROPOSITION 2. *Constraint (6) is equivalent to the following system of linear inequalities.*

$$y_i^t \leq \alpha_i(\boldsymbol{h}_{i,0}^{\sigma})(\gamma_i^t - \tilde{z}_{i\sigma(1)}^t) + \sum_{k\in[M]} \alpha_i(\boldsymbol{h}_{i,k}^{\sigma})(\tilde{z}_{i\sigma(k)}^t - \tilde{z}_{i\sigma(k+1)}^t) \qquad \text{for } \sigma \in \Omega \tag{7}$$

where $\tilde{z}_{i\sigma(M+1)}^t = 0$, $\tilde{z}_{i\sigma(k)}^t = z_{i\sigma(k)}^t$ *if* $\sigma(k) \notin I_i$, *and* $\tilde{z}_{i\sigma(k)}^t = \gamma_i^t - z_{i\sigma(k)}^t$ *if* $\sigma(k) \in I_i$.

3.1.2. The Final Formulation Using the linear constraints (3), (4), and (7) and the exponential cone constraint (5) to represent (CHOICE), we obtain an MIECP formulation of problem (HAP) as follows:

$$\begin{aligned} \max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{\rho},\boldsymbol{\gamma},\boldsymbol{z}} \quad & \frac{1}{T}\sum_{t\in[T]}\sum_{i\in[N]} r_i y_i^t \\ \text{s.t.} \quad & \boldsymbol{x}^t \in \{0,1\}^N \cap \mathcal{X} \text{ and } \boldsymbol{x} \in \mathcal{P} && \text{for } t \in [T] \\ & \rho^t + \sum_{i\in[N]} y_i^t = 1 && \text{for } t \in [T] \\ & (3),\ (4),\ (5),\ (7) && \text{for } t \in [T] \text{ and } i \in [N]. \end{aligned} \tag{CONIC}$$

THEOREM 1. *(CONIC) is a mixed-integer exponential cone formulation of problem (HAP).*

REMARK 1. In this remark, we consider the case where the memory length is at most two. In this case, we can derive the tightest convex extension, also known as the *convex envelope*, of $\alpha_i(\cdot)$. Using the convex envelope instead of the natural continuous extension, we obtain the following tighter formulation than the one given by constraint (5) for $M = 2$. If $\boldsymbol{\beta}_i$ are all positive or negative, we have

$$y_i^t \geq \alpha_i(\mathbf{0})(\gamma_i^t - z_{i1}^t - z_{i2}^t) + \alpha_i(\boldsymbol{e}_1)z_{i1}^t + \alpha_i(\boldsymbol{e}_2)z_{i2}^t \tag{8a}$$

$$y_i^t \geq \alpha_i(\boldsymbol{e}_1)(\gamma_i^t - z_{i2}^t) + \alpha_i(\boldsymbol{e}_2)(\gamma_i^t - z_{i1}^t) + \alpha_i(\mathbf{1})(z_{i1}^t + z_{i2}^t - \gamma_i^t), \tag{8b}$$

and if $\boldsymbol{\beta}_i$ are mixed, the convex envelope is

$$y_i^t \geq \alpha_i(\mathbf{0})(\gamma_i^t - z_{i1}^t) + \alpha_i(\boldsymbol{e}_1)(z_{i1}^t - z_{i2}^t) + \alpha_i(\mathbf{1})z_{i2}^t \tag{9a}$$

$$y_i^t \geq \alpha_i(\mathbf{0})(\gamma_i^t - z_{i2}^t) + \alpha_i(\boldsymbol{e}_2)(z_{i2}^t - z_{i1}^t) + \alpha_i(\mathbf{1})z_{i1}^t \tag{9b}$$

where $\boldsymbol{e}_m \in \{0,1\}^2$ is the unit vector with one as the m^{th} coordinate and zero everywhere else, and $\mathbf{1} = (1,1)$. For $M = 1$, constraints (8) and (9) reduce to $y_i^t \geq \big(\alpha_i(1) - \alpha_i(0)\big)z_{i1}^t + \alpha_i(0)\gamma_i^t$. In addition to the tightness, another advantage of constraint (8) and (9) over the conic constraint (5) is that they are linear inequalities. Therefore, replacing (5) in (CONIC) with (8) and (9), we obtain an MILP formulation of (HAP). This allows us to solve (HAP) using a commercial MILP solver—such as `Gurobi`—which has advantages over its conic counterparts in terms of practical scalability and numerical stability, as illustrated in Section 6.1. $\square$

Remark 2. For each period $t \in [T]$ and product $i \in [N]$, constraint (7) requires $M!$ linear inequalities, which grow factorially with M and quickly become intractable. To overcome this scalability issue, we develop a cutting-plane algorithm for solving the MIECP formulation. The algorithm leverages `Gurobi`'s `Callback` routine to implement constraint (7) as lazy constraints. Further implementation details are provided in EC.5.3. □

Remark 3. We note that our formulation technique extends to a broad class of attraction value functions. In particular, it accommodates any function $\alpha_i(\cdot)$ that can be written as the composition of a univariate convex function and an affine function. Clearly, this includes the exponential-of-linear function as a special case. □

We conclude this section with a discussion of how our formulation relates to existing formulations. Recently, a large amount of research has focused on developing strong formulations for binary linear fractional programming problems and MNL-based assortment planning problems (e.g. Sen et al. 2018, Mehmanchi et al. 2019, Atamtürk and Gómez 2020, Kılınç-Karzan et al. 2023, He et al. 2024, Chen et al. 2025). To apply these results in solving problem (HAP), we need to linearize the nonlinear functions appearing in denominators and numerators—specifically, the attraction value functions. A prevalent linearization technique replaces each attraction value function by its multilinear extension, called the Fourier Expansion in O'Donnell (2014), and then linearizes the resulting multilinear functions using the recursive McCormick relaxation (McCormick 1976, Khajavirad 2023). We combine this linearization strategy with binary linear fractional programming formulation techniques to obtain an exact formulation of problem (HAP), which we refer to as the multilinear-extension-based formulation. The complete derivation is provided in the ecompanion EC.5.1. Later, in Section 6.1, a computational experiment shows that our formulation (Conic) significantly outperforms the multilinear-extension-based formulation.

Our problem setting is related to the logit-based share-of-choice product design (SOCPD) studied in Akçakuş and Mišić (2025), which aims to maximize the share of K customer types by selecting n binary design attributes for a single product, that is,

$$\max_{\boldsymbol{a} \in \mathcal{A} \subseteq \{0,1\}^n} \quad \sum_{k=1}^{K} \lambda_k \cdot \frac{\exp(u_k(\boldsymbol{a}))}{1 + \exp(u_k(\boldsymbol{a}))},$$

where for each customer type k, $u_k(\boldsymbol{a})$ is an affine utility function for the product, the fraction term models the purchase probability of a customer of type k, and λ_k is the fraction of customers who belong to type k. Our model shares some similarities with SOCPD since the exponential-affine attraction value functions appear in both problems. However, there are two major differences. First, since we consider assortments of multiple products, our formulation (Conic) requires both convex and concave extensions of the attraction value function to achieve exactness. In contrast, since SOCPD treats only one single product, it suffices to use the natural convex extension of the exponential-affine function in that case. Second, in our model, the attraction value function is multiplied by an assortment decision variable to model whether the attractiveness of a product is activated in the current period. This requires us to adapt the perspectification trick used in Akçakuş and Mišić (2025) to allow for nonlinear scaling, as discussed in Section 3.1.1.

3.2. Optimality of Revenue-Ordered Policies

A prevalent strategy for solving assortment optimization is to construct Revenue-Ordered (RO) assortments, which is an assortment including products with revenue higher than a threshold. Specifically, RO structures and their variants have been shown to be optimal for solving assortment optimization problems under the MNL choice model (Talluri and Van Ryzin 2004), sequential MNL choice models (Gao et al. 2021), MNL choice models with network effects (Wang and Wang 2017), etc. Inspired by the simplicity and optimality of RO structures, we would like to understand how it works in our problem.

We find that when there are no cross-product and cross-period constraints, if the history-dependent effect is non-negative, that is, $\boldsymbol{\beta} \geq 0$, the optimal assortment in each period is an RO assortment, and the revenue threshold increases when T increases. Such an assortment planning can be obtained by a *sequential revenue-ordered* policy. The sequential RO policy is obtained by sequentially selecting the best RO assortment based on the currently updated product utility and maximizing revenue in the current period. This policy is adaptive while ignoring the impact of the current assortment decision on future revenue. Details of the sequential RO policy are shown in Algorithm 1. Because there are no cross-product and cross-period constraints, the computation time required to find the optimal RO assortment is $\mathcal{O}(N)$ in each period (Talluri and Van Ryzin 2004, Liu and Van Ryzin 2008). Thus, the total computation time of the sequential RO policy is $\mathcal{O}(NT)$.

Algorithm 1 Sequential revenue-ordered policy of (HAP)

1: Input: $\boldsymbol{r}, \{\beta_i^0\}_{i\in[N]}, \boldsymbol{\beta}, M, N, T$
2: **for** $t = 1, 2, \ldots, T$ **do**
3: Update history assortments: $\boldsymbol{x}^{t-1}, \ldots, \boldsymbol{x}^{t-M}$, and set $\boldsymbol{x}^{t-m} = \mathbf{0}$ if $t - m \leq 0$.
4: Compute the best revenue-ordered assortment $\boldsymbol{x}^t = \underset{\boldsymbol{z}\in\{0,1\}^N}{\operatorname{argmax}} \sum_{i\in[N]} r_i \pi_i^t(\boldsymbol{z}, \boldsymbol{x}^{t-1}, \ldots, \boldsymbol{x}^{t-M})$
5: **end for**
6: Output: $(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^T)$

We are now in a position to state the main result of this subsection, Theorem 2, characterizing conditions on which the sequential RO policy is optimal.

THEOREM 2. *Assume the absence of cross-product and cross-period constraints. Then, the sequential-revenue-ordered policy solves* (HAP) *if the history-dependent effects are non-negative, that is,* $\boldsymbol{\beta} \geq \mathbf{0}$.

For the case of negative history-dependent effects, which is the computationally intractable regime of our model, as we argued in Proposition 1, the sequential policy fails to solve our model. However, the RO structure still exists under the negative and even mixed history-dependent effects and appears in a different form. For an optimal assortment planning of model (HAP), we argue that the set of products that are offered across all periods has an RO structure.

PROPOSITION 3. *Assume the absence of cross-product and cross-period constraints. Then, there exists an optimal assortment* $\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^T)$ *of* (HAP) *such that* $\left\{ i \in [N] \,\middle|\, \sum_{t \in T} x_i^t \geq 1 \right\}$ *is revenue-ordered.*

4. Cyclic Policies: Optimality and Computation

In this section, we consider the situation in which a firm plans assortments for a long period. We establish the asymptotic optimality of cyclic policies in Section 4.1 and formulate the cyclic policy problem as an MIECP in Section 4.2. In Section 4.3, we characterize conditions under which the length of optimal cyclic policies is the memory length plus one. In Section 4.4, we show that such conditions lead to a tighter MILP formulation for finding optimal $(M+1)$-cyclic policies.

4.1. Optimality of Cyclic Policies

The assortment planning problem with an infinite horizon can be formulated as follows. The firm can select any assortment sequence (or policy) $\boldsymbol{x} = \{\boldsymbol{x}^t\}_{t \in \mathbb{Z}_+}$ with elements in $\{0,1\}^N$. As before, the firm may face cross-product and cross-period constraints. Let $\mathcal{X}$ (resp. $\mathcal{P}^\infty$) denote the feasible set of cross-product (resp. cross-period) constraints. Then, the firm aims to find an optimal assortment sequence for the following problem:

$$\sup_{\boldsymbol{x}} \left\{ \lim_{T \to \infty} \frac{1}{T} \sum_{t \in [T]} \sum_{i \in [N]} r_i \pi_i^t(\boldsymbol{x}^t, \ldots, \boldsymbol{x}^{t-M}) \,\middle|\, \boldsymbol{x}^t \in \mathcal{X} \cap \{0,1\}^N \text{ for } t \in \mathbb{Z}_+, \ \boldsymbol{x} \in \mathcal{P}^\infty \right\}. \tag{INFTY}$$

An assortment policy $\boldsymbol{x} = \{\boldsymbol{x}^t\}_{t \in \mathbb{Z}_+}$ is *cyclic* if there exists a positive integer L such that $\boldsymbol{x}^{t+L} = \boldsymbol{x}^t$ for all $t \in \mathbb{Z}_+$. The smallest L for which this holds is called the cycle length of $\boldsymbol{x}$. A cyclic policy $\boldsymbol{x}$ can be represented by a finite sequence of assortments $(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^L)$, where L is the cycle length of $\boldsymbol{x}$. In contrast, whenever we say that a finite sequence of assortments $(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^L)$ is cyclic, we are referring to the policy in which this finite sequence of assortments is repeated infinitely often.

For a given cycle of length L, we are interested in finding a cyclic policy of length L that maximizes the long-term average revenue. This is equivalent to searching for a sequence of assortments $(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^L)$ of length L that maximizes the average revenue by repeating the sequence. Such a sequence can be defined as an optimal solution of a variant of (HAP). To define the variant, we modify the attraction value function so that it depends on past assortments in the cycle. Given a position t in the cycle of length L, we use a function $\tau(m|t)$ to track position m within the memory length M:

$$\tau(m|t) = \begin{cases} (t + L - m \bmod L) & \text{if } (t + L - m \bmod L) > 0 \\ L & \text{if } (t + L - m \bmod L) = 0. \end{cases}$$

The attraction value of product i at position t can be specified as follows:

$$\alpha_i^t(x_i^1,\ldots,x_i^L) = \exp\left(\beta_i^0 + \sum_{m\in[M]} \beta_i^m x_i^{\tau(m|t)}\right).$$

A cyclic policy is called an L*-cyclic policy* if it is optimal for the following variant of problem (HAP):

$$\max\left\{\frac{1}{L}\sum_{t\in[L]}\sum_{i\in[N]} \frac{r_i x_i^t \alpha_i^t(x_i^1,\ldots,x_i^L)}{1+\sum_{j\in[N]} x_j^t \alpha_j^t(x_i^1,\ldots,x_i^L)} \;\middle|\; \boldsymbol{x}^t \in \mathcal{X}\cap\{0,1\}^N \text{ for } t\in[L],\ \boldsymbol{x}\in\mathcal{P}^L\right\}, \tag{CYCLE}$$

where $\mathcal{X}$ denotes cross-product constraints and $\mathcal{P}^L$ denotes cross-period constraints. In addition, we assume that a finite sequence of assortments satisfying $\mathcal{P}^L$ can be repeated infinitely often to obtain a cyclic policy that satisfies the cross-period constraint $\mathcal{P}^\infty$ in (INFTY).

The main result of this subsection relates to problems (INFTY) and (CYCLE). To establish this result, we represent (INFTY) as a directed graph, called the *assortment graph*. Each node of the graph denotes a list of M assortments, indexed from the latest to the earliest assortments in memory. Each arc represents a memory transition driven by an assortment decision. Specifically, an arc starting at a node should end at a node that drops the earliest assortment of the starting node and adds a new assortment. Hence, each arc encodes an assortment decision in a period, and its weight is defined as the single-period revenue generated by the new assortment. With such a graph, we can show that (INFTY) is equivalent to finding the maximum mean cycle of the assortment graph (Karp 1978), whose cycle length is denoted as L^*. Given L^*, we can use (CYCLE) to compute the cyclic policy. We formalize this observation as the following theorem.

THEOREM 3. *Given an instance of* (INFTY)*, there exists a positive integer L^* such that L^*-cyclic policy is optimal to* (INFTY).

4.2. Finding Optimal Cyclic Policies

Formulation techniques developed for solving (HAP) can be used to obtain a mixed-integer exponential cone formulation for (CYCLE). Since the main discrepancy between the two models is the definition of attraction value functions, we only need to modify constraints in (CONIC) that involve attraction value functions. In particular, we replace x_i^{t-m} in (4) by $x_i^{\tau(m|t)}$ and $\alpha_i(\cdot)$ in (7) by $\alpha_i^t(\cdot)$:

$$\max\{0,\ \rho_U^t x_i^{\tau(m|t)} + \gamma_i^t - \rho_U^t\} \le z_{im}^t \le \min\{\gamma_i^t, \rho_U^t x_i^{\tau(m|t)}\} \qquad \text{for } m\in[M] \tag{10a}$$

$$y_i^t \le \alpha_i^t(\boldsymbol{w}_{i,0}^\sigma)(\gamma_i^t - \tilde{z}_{i\sigma(1)}^t) + \sum_{k\in[M]} \alpha_i^t(\boldsymbol{w}_{i,k}^\sigma)(\tilde{z}_{i\sigma(k)}^t - \tilde{z}_{i\sigma(k+1)}^t) \qquad \text{for } \sigma\in\Omega, \tag{10b}$$

where $\boldsymbol{w}_{i,k}^\sigma \in \{0,1\}^L$, and its $\tau(m|t)^{\text{th}}$ element equals the m^{th} element of $\boldsymbol{h}_{i,k}^\sigma$ defined in (7) and others are zero. This replacement yields the following mixed-integer exponential cone formulation of (CYCLE):

$$\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{\rho},\boldsymbol{\gamma},\boldsymbol{z}} \quad & \frac{1}{L}\sum_{t\in[L]}\sum_{i\in[N]} r_i y_i^t \\
\text{s.t.} \quad & \boldsymbol{x}^t \in \mathcal{X}\cap\{0,1\}^N \text{ and } \boldsymbol{x}\in\mathcal{P}^L && \text{for } t\in[L] \\
& \rho^t + \sum_{i\in[N]} y_i^t = 1 && \text{for } t\in[L] \\
& (3)\ (5)\ (10) && \text{for } t\in[L] \text{ and } i\in[N].
\end{aligned} \tag{CYCLE-CONIC}$$

(CYCLE-CONIC) has its own value because it is an efficient way to generate a feasible assortment planning with a large planning horizon. Especially, the retailer may pre-determine a cycle length L based on experience or industry knowledge and use (CYCLE-CONIC) to derive the optimal policy under a given L.

4.3. Cycle Length Characterization

It is intuitive that, in order to avoid the negative impact on a product's utility if the product exhibits a satiation effect, a product should not be offered consecutively. In particular, we say that a cross-period constraint on a policy $\boldsymbol{x}$ is *non-overlapping* if $M+1$ adjacent assortments $(\boldsymbol{x}^t, \boldsymbol{x}^{t+1}, \ldots, \boldsymbol{x}^{t+M})$ are non-overlapping, that is, for each product $i \in [N]$, $x_i^t + x_i^{t+1} + \cdots + x_i^{t+M} \leq 1$ for every $t \in \mathbb{Z}_+$. This non-overlapping condition is common in assortment optimization literature (Liu et al. 2020, Chen et al. 2023b). For instance, Li et al. (2024) identify a similar non-overlapping condition on customer orders to plan assortments for a local warehouse. Under the non-overlapping condition, any cyclic policy with length $L \leq M$ must have empty sets in some periods. We do not consider such cyclic policies. For cyclic policies with length $L > M$, we find that this condition allows us to relate the length of an optimal cyclic policy for solving (INFTY) to the customers' memory length as follows.

THEOREM 4. *Assume that $M < N$. An $(M+1)$-cyclic policy is optimal for problem* (INFTY) *if the cross-period constraint is non-overlapping.*

We remark that Theorem 4 holds for mixed satiation-addiction effects, because its proof replies on a graph whose structure unaffected by signs of $\boldsymbol{\beta}$.

The condition $M < N$ guarantees that there at least exists a non-overlapping assortment policy without empty assortments. Without the non-overlapping condition, the optimal cycle length varies as the parameters change. In particular, a slight fluctuation in the input parameters, like the base utility of a product, could change the optimal cycle length. We provide an example to illustrate this situation in EC.4.4.

The base utility and history-dependent effect are estimated from real data and lie in confidence intervals. Hence, given the fact that the optimal cycle length is sensitive to input parameters, finding a cycle length with robust performance may be more critical than characterizing the optimal cycle length in practice. A positive result is that although the $(M+1)$-cyclic policy is not necessarily optimal for (INFTY) without the non-overlapping constraint, it still provides a good approximation, as illustrated in the following example.

EXAMPLE 1. We numerically test the performance of the $(M+1)$-cyclic policy under satiation effects with $N = 30$ and $T \in \{10, 30, 50\}$. We generate revenue r_i and base utility β_i^0 from uniform distributions with ranges $[1, 10]$ and $[-1, 1]$, respectively. To model different levels of satiation, we sample β_i^m from $U[-1, 0]$ for weak (W) effects and from $U[-2, -1]$ for strong (S) effects. We define the revenue gap between the $(M+1)$-cyclic policy and the best feasible solution as $100\% \times \frac{R_{\mathsf{fea}} - R_{\mathsf{cycle}}}{R_{\mathsf{fea}}}$, where R_{cycle} is the revenue achieved by the $(M+1)$-cyclic policy and R_{fea} is the revenue of the best feasible solution obtained by (CONIC) within 7200 seconds. Table 1 records the average gap of five instances of each parameter con-

Table 1 **Revenue gap of $(M+1)$-cyclic policies under satiation effects with $N=30$ and no constraints**

$\boldsymbol{\beta}$	T	$M=1$ (%)	$M=2$ (%)	$\boldsymbol{\beta}$	T	$M=1$ (%)	$M=2$ (%)
W	10	0.51	1.69	S	10	0.39	4.45
W	30	0.16	0.60	S	30	-0.06	-0.50
W	50	0.08	0.52	S	50	-0.52	0.46

figuration. A negative gap indicates that the revenue generated by the $(M+1)$-cyclic policy is higher than the revenue generated by the best feasible solution achieved within 7200s. Overall, the $(M+1)$-cyclic policy performs well. Moreover, as the length of the planning horizon increases, the revenue gap decreases.

4.4. A Strong Formulation for Finding Non-overlapping $(M+1)$-Cyclic Policies

In this subsection, we propose a bound-free formulation for the $(M+1)$-cyclic policy, which is provably tighter than its conic counterpart (CYCLE-CONIC) whose computational performance depends on the bounds (ρ_L^t and ρ_U^t) of the no-purchase probability variable ρ^t.

From the non-overlapping constraint $x_i^t + x_i^{\tau(1|t)} + \cdots + x_i^{\tau(M|t)} \leq 1$, we have:

$$x_i^t \alpha_i^t(x_i^1, \ldots, x_i^L) = x_i^t \exp\left(\beta_i^0 + \sum_{m\in[M]} \beta_i^m x_i^{\tau(m|t)}\right) = x_i^t \exp(\beta_i^0) \qquad \text{for } t \in [M+1].$$

Let $u_i = \exp(\beta_i^0)$. (CYCLE) is equivalent to

$$\begin{aligned} \max \quad & \frac{1}{M+1} \sum_{t\in[M+1]} \sum_{i\in[N]} \frac{r_i u_i x_i^t}{1+\sum_{j\in[N]} u_j x_j^t} \\ \text{s.t.} \quad & x_i^t + \sum_{m\in[M]} x_i^{\tau(m|t)} \leq 1 && \text{for } i \in [N] \\ & \boldsymbol{x}^t \in \mathcal{X} \cap \{0,1\}^N && \text{for } t \in [M+1]. \end{aligned} \tag{(M+1)-CYCLIC}$$

Note that this reformulation reveals a connection between (CYCLE) and the assortment problem studied in Caro et al. (2014) in which a retailer must decide, in advance, the release date of each product in a given collection over a selling season. More specifically, when $M=1$, ((M+1)-CYCLIC) is exactly the case where a retailer faces two planning horizons and sells products with a single period life-cycle, and, in each period, customers make a purchase decision on offered products according to the MNL model.

Next, we introduce variables for building a strong formulation of ((M+1)-CYCLIC). For each period $t \in [M+1]$, we use ρ^t and $u_i \cdot \gamma_i^t$ to represent the no-purchase probability and choice probability of product i, respectively,

$$\rho^t = \frac{1}{1+\sum_{j\in[N]} u_j x_j^t} \quad \text{and} \quad \gamma_i^t = \frac{x_i^t}{1+\sum_{j\in[N]} u_j x_j^t} \qquad \text{for } i \in [N]. \tag{CC}$$

Note that this is known as the Charnes-Cooper transformation (Charnes and Cooper 1962). For each period $t \in [M+1]$, it maps all possible assortments $\boldsymbol{x}^t \in \{0,1\}^N$ into all possible choice probabilities $(\rho^t, u_1\gamma_1^t, \ldots, u_N\gamma_N^t)$, where $(\rho^t, \boldsymbol{\gamma}^t)$ belongs to

$$\rho^t + \sum_{j\in[N]} u_j \gamma_j^t = 1 \quad \text{and} \quad \gamma_j^t \in \{0, \rho^t\} \text{ for } j \in [N]. \tag{11}$$

Using these relations, we are ready to present a new formulation of ((M+1)-CYCLIC):

$$
\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{\rho},\boldsymbol{\gamma},\boldsymbol{\Gamma}} \quad & \frac{1}{M+1}\sum_{t\in[M+1]}\sum_{i\in[N]} r_i u_i \gamma_i^t \\
\text{s.t.} \quad & x_i^t + \sum_{m\in[M]} x_i^{\tau(m|t)} \le 1 \text{ and } \boldsymbol{x}^t \in \mathcal{X}\cap\{0,1\}^N \quad \text{for } i\in[N] \text{ and } t\in[M+1] && \text{(12a)}\\
& \rho^t + \sum_{i\in[N]} u_i\gamma_i^t = 1 \text{ and } 0\le \gamma_i^t \le \rho^t \quad \text{for } i\in[N] \text{ and } t\in[M+1] && \text{(12b)}\\
& x_i^t = \gamma_i^t + \sum_{j\in[N]} u_j \Gamma_{ij}^t \quad \text{for } i\in[N] \text{ and } t\in[M+1] && \text{(12c)}\\
& \max\{0,\gamma_i^t+\gamma_j^t-\rho^t\} \le \Gamma_{ij}^t \le \min\{\gamma_i^t,\gamma_j^t\} \quad \text{for } i\in[N],\, j\in[N]\setminus\{i\}, \text{ and } t\in[M+1] && \text{(12d)}\\
& \Gamma_{ii}^t = \gamma_i^t \quad \text{for } i\in[N] \text{ and } t\in[M+1]. && \text{(12e)}
\end{aligned}
$$

The constraint (12b) is obtained from relaxing the constraints from (11). The constraint (12c) is derived from the following equality, with the variable γ_i^t replacing the first fractional term and a new variable Γ_{ij}^t replacing each fractional term in the summation,

$$x_i^t = \frac{x_i^t}{1+\sum_{k\in[N]} u_k x_k^t} + \sum_{j\in[N]} u_j \frac{x_i^t x_j^t}{1+\sum_{k\in[N]} u_k x_k^t} \qquad \text{for } t\in[M+1].$$

To obtain (12d), we use the no-purchase variable ρ^t to scale the following McCormick envelope of $x_i^t x_j^t$ (McCormick 1976):

$$\max\{0, x_i^t + x_j^t - 1\} \le x_i^t x_j^t \le \min\{x_i^t, x_j^t\} \qquad \text{for } i\in[N], j\in[N]\setminus\{i\}.$$

The last constraint (12e) is obtained by observing that the relation $x_i^t x_i^t = x_i^t$ holds for binary variables. Unlike the formulation (CYCLE-CONIC), this formulation does not rely on bounds of the no-purchase probability variables. More specifically, in the derivation of (CYCLE-CONIC), we use $\rho^t x_i^t$ as a nonlinear scaling factor, demanding a linearization using bounds of ρ^t. In the development of linear inequalities in (12), we avoid using a nonlinear scaling factor and, instead, directly use ρ^t as a scaling variable to derive (12d).

The following result shows this bound-free formulation is theoretically tighter than the formulation (CYCLE-CONIC). One of the most important properties of an mixed integer program (MIP) formulation is the strength of its natural continuous relaxation that is obtained by ignoring the integrality constraints. This is important because a tighter continuous relaxation often indicates a faster convergence of the branch-and-bound algorithm on which most commercial MIP solvers are built (Vielma 2015). Since (HAP) has a maximization objective, a tighter formulation has a smaller continuous relaxation objective value. Let $Z_{\text{BOUND-FREE}}$ (resp. $Z_{\text{CYCLE-CONIC}}$) be the optimal objective value of the natural continuous relaxation of formulation (12) (resp. (CYCLE-CONIC)). The two objective values have the following relationship.

THEOREM 5. *$Z_{\text{BOUND-FREE}} \le Z_{\text{CYCLE-CONIC}}$ if the cross-period constraint is non-overlapping.*

5. Case Study

This section presents a case study using transaction data from a corporation cafeteria. We first estimate the history-dependent effect and then use the results to design assortments. Our estimations indicate that customers have a significant but short-term satiation effect. Furthermore, our findings suggest that including history-dependent effects in choice models will lead to greater assortment varieties and higher revenue.

5.1. Data Description and Estimation

We cooperated with a Chinese company that provides catering services to over $4,000$ corporation cafeterias and were given access to a dataset for one cafeteria that mainly serves workday luncheons for customers who work nearby. The dataset includes $27,144$ transactions for 53 main dishes, covering 110 workdays from July to December 2021. The data for each transaction includes the time, dish, and quantity, as well as a card ID if the dish was purchased with a membership card. The dataset also includes information about the price, main ingredient, and flavor of each dish. In what follows, we use the terms dish and product and also the terms menu and assortment interchangeably.

Our dataset indicates that 84.8% of the transactions came from 418 members, which generated 84.5% of the total revenue. Thus, we focus on the members' transactions because they are frequent visitors and generate most of the revenue. Our dataset also reveals that the probability of a member dining in the cafeteria more than once for two, three, or four consecutive days was 72.5%, 87.7%, and 91.2%, respectively. This suggests that customers' dish selections could be influenced by past menus.

We predict customers' choices using the MNL model with history-dependent effects. The utility function for product i on day t is

$$U_i^t = \boldsymbol{\theta}^T X_i + \theta_r r_i + \sum_{m\in[M]} x_i^{t-m}\beta^m + \xi_i^t,$$

where X_i denotes covariates that include a fixed effect and the main ingredient and flavor of product i; $r_i \in [0,1]$ is a normalized price; $\boldsymbol{\theta}$ and θ_r are coefficients of X_i and price, respectively; $\boldsymbol{\theta}^T X_i + \theta_r r_i$ is the base utility β_i^0 defined in Section 2; and ξ_i^ts are i.i.d. standard Gumbel random variables. The history-dependent effect is captured by the term $\sum_{m\in[M]} x_i^{t-m}\beta^m$, in which M is the memory length. Our model is a standard MNL model if $M=0$. Note that we set the same history-dependent coefficient for all products because not all products have consecutive offering records, and therefore, we cannot estimate the product-wise history-dependent effect.

We utilize members' transactional data to calibrate the MNL choice model with the utility function presented above. We split the transaction data into two sets: data from July to October for training and data from November to December for test. We first use five-fold cross-validation on the first data set to estimate the utility function, and then we optimize weekly menus based on the estimators. Next, we conduct the five-fold cross-validation again, this time on the second data set, to select the value of M with the best

Table 2 **Estimators and cross-validation results under different memory length**

	Estimators					Out-of-sample Performance	
M	θ_r	β^1	β^2	β^3	β^4	χ^2	Log-likelihood
0	-2.558^{***}	-	-	-	-	0.0860	-1.7796
	(0.0767)						
1	-2.550^{***}	-0.475^{***}	-	-	-	0.0828 (3.67%)	-1.7769 (0.15%)
	(0.0767)	(0.0498)					
2	-2.520^{***}	-0.493^{***}	-0.366^{***}	-	-	0.0822 (4.38%)	-1.7754 (0.24%)
	(0.0767)	(0.0499)	(0.0568)				
3	-2.540^{***}	-0.492^{***}	-0.358^{***}	-0.289^{***}	-	0.0819 (4.71%)	-1.7749 (0.27%)
	(0.0768)	(0.0500)	(0.0568)	(0.0708)			
4	-2.540^{***}	-0.495^{***}	-0.361^{***}	-0.286^{***}	-0.146^{***}	0.0826 (3.91%)	-1.7753 (0.25%)
	(0.0768)	(0.0500)	(0.0569)	(0.0709)	(0.0647)		

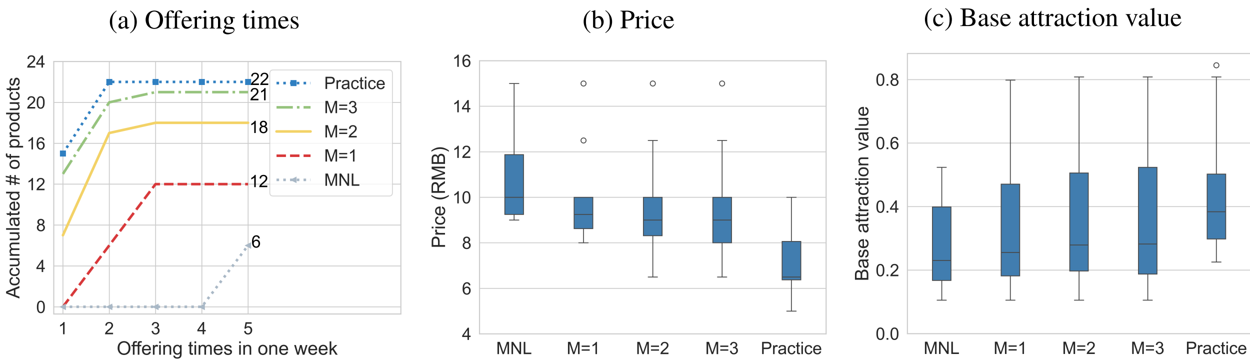
Note: The numbers in parentheses under the estimators report the standard errors of the estimations. "-" indicates that a model does not include the corresponding predictor. $***p < 0.001$. The columns "χ^2" and "Log-likelihood" show the average value of the five-fold cross-validation (normalized by sample size) followed by a percentage in parentheses, which is the relative improvement of each model with a history-dependent effect compared to the standard MNL model.

out-of-sample performance, and we set this as the "ground truth" model for evaluation, which is a common approach in policy evaluation (Cao et al. 2022). To address the fact that our dataset lacks no-purchase records, we employ the expectation-maximization (EM) approach, a widely used technique to deal with censored data (van Ryzin and Vulcano 2017, Şimşek and Topaloglu 2018), to calibrate our model.

Table 2 summarizes the estimation and five-fold validation results for $M \in \{0,1,2,3,4\}$, using the first data set. Column θ_r presents price coefficient estimators, and columns β^1 to β^4 give history-dependent effects. The columns labeled χ^2 and Log-likelihood provide the average out-of-sample value in the five-fold cross-validation (normalized by sample size). The percentage values in parentheses indicate the relative improvement of each model with a history-dependent effect compared to the standard MNL model. Table 2 shows significant negative history-dependent effects, and the scale of the effects decreases as the time interval between the current and historical offerings increases. The estimation results are consistent for different values of M. For instance, β^1 is consistently about -0.49 for all M. This suggests that the attraction value of a product will shrink by about $\exp(-0.49) = 61\%$ if it was offered yesterday. In other words, if a product is sold on two consecutive days, we need to lower its price by at least RMB $(0.492/2.540) \times 15 \approx 2.9$ when $M = 3$ to compensate for the negative history-dependent effect, where we normalize prices by the maximum price RMB 15 in the estimation. These results demonstrate the importance of considering history-dependent effects in assortment planning.

The average out-of-sample validation results are reported in the last two columns of Table 2. They suggest that history-dependent effects can improve prediction power (i.e., lower χ^2 and higher log-likelihoods). When $M = 3$, the model presents the lowest χ^2 and the highest log-likelihood. This result suggests that a small memory length ($M = 3$) can capture the history-dependent effect in customers' choice.

Figure 3 **Information of products offered in assortments obtained by different models**



5.2. Assortment Planning

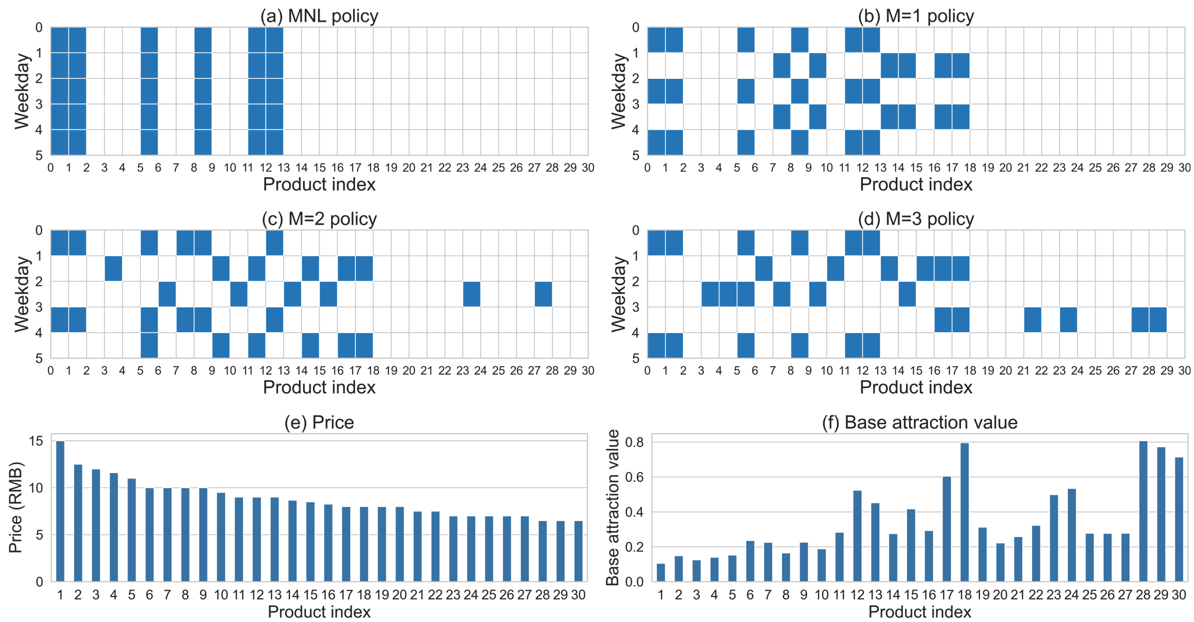
In this section, we use the formulation proposed in Section 3.1 to optimize weekly menus using the above estimators. We design weekly workday menus because the cafeteria does not frequently operate on weekends, and in practice, the cafeteria indeed plans for a week. We use the estimated parameters in Table 2 as the input for our conic models. To keep the menu sizes comparable, we impose a daily cardinality constraint of 6, which is the median assortment size in the dataset.

5.2.1. Assortment Patterns Figure 3 shows the accumulated number of products for different offering times, the prices, and the base attraction values obtained from the assortments under the MNL model, under history-dependent MNL models with different memory lengths, and under an assortment planning used in practice. The current practice assortment was randomly chosen from the second test data set.

As shown in Figure 3a and 3b, the MNL-based model tends to offer fewer products with higher prices. Moreover, the number of products that are offered increases with the memory length M, because it is beneficial to offer more products and avoid repeating the same product within the memory length when satiation effects last for a long period. When the memory length increases, the assortments include more products with lower prices and higher base attraction values, as shown in Figure 3b and 3c.

Figure 4 presents assortment details for examining how the memory length affects assortment patterns. Figure 4 (a)-(d) show the assortments obtained from MNL-based models with $M \in \{0, 1, 2, 3\}$. The horizontal axes represent products indexed from the highest to the lowest price. The vertical axes represent the days of a week. The products offered are colored blue. The subfigures show that the interval between two adjacent offerings of the same dish grows when M increases, and optimal assortments represent a cyclic-style structure. The interval for two adjacent offerings of most products is the memory length M, which suggests that the cycle length is $M+1$, aligning with the $(M+1)$-cyclic policy in Section 4. Moreover, we observe that products with higher prices or lower base attraction values are more sensitive to the satiation effect and thus will be offered at most once within the span of the memory length. For instance, the high-price products 1 to 5 strictly repeat once in intervals of M days to avoid the satiation effect, possibly

Figure 4 Assortment details of four different policies



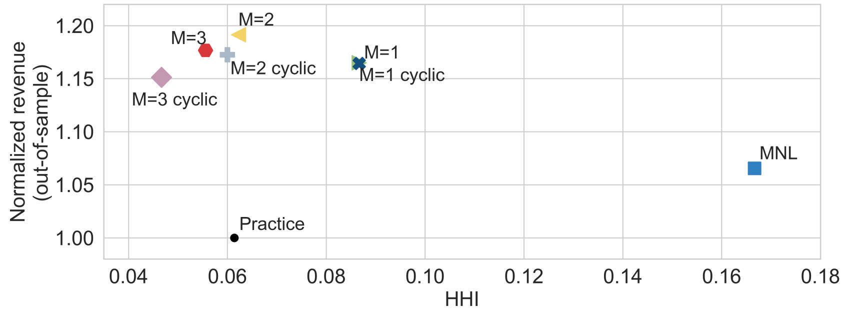
Note. (a)-(d) show the assortments of four polices. The horizontal axes represent products indexed. The vertical axes represent the days of a week. The products offered are colored blue. (e) and (f) show the price and base attraction value of each product.

because the revenue loss caused by the utility deterioration will be scaled up by the high price. On the other hand, products with relatively high base attraction value may possibly be offered more than once within the memory period, e.g., product 17,18. The price and base attraction value for each product are shown in Figure 4 (e) and (f) for reference.

5.2.2. Performance Comparison In this section, we focus on two metrics to evaluate the assortment performance: variety and revenue, as these are concerns for the corporation cafeteria. We adopt the Herfindahl-Hirschman Index (HHI), a measure of market concentration (Rhoades 1993), to quantify the variety of an assortment planning. The HHI value is $\sum_{i\in[n]}(\frac{h_i}{H})^2$, where n is the total number of products, h_i is the number of times product i is offered, and $H = \sum_{i\in[n]} h_i$. A low HHI value reflects a large variety. To compute the out-of-sample revenue, we randomly sample 20 subsets from the second data set. In each subset, we estimate the "ground truth" model with $M = 2$ that was selected by cross-validation on the second data set and calculate the revenue based on the "ground truth" demand model. The average revenue across the 20 data sets is used as the out-of-sample revenue to avoid random noise. Besides the assortments discussed in Section 5.2.1, we additionally compute assortments obtained by the $(M+1)$-cyclic policy with non-overlapping constraints that is presented in Section 4.

Figure 5 illustrates the relationship between the variety and the revenue of an assortment plan. The horizontal axis represents the HHI value, and the vertical axis shows the average normalized revenue. We normalize the out-of-sample revenue of different assortment plans with the revenue of current practice as

Figure 5 **Assortment variety and average revenue in the out-of-sample test**



Note. The horizontal axis represents the Herfindahl-Hirschman Index value of each policy. A small HHI value means a wide variety. The vertical axis shows the average revenue of a policy across 20 test sets. We normalize the revenue of the current practice to one.

a benchmark. Generally, assortments with a low HHI value can help avoid the negative impacts of satiation effects and maintain high product utility. Figure 5 shows that the assortments generated by our model ($M > 0$) have a lower HHI and higher revenue than the assortments obtained with the MNL model. For instance, the average revenue obtained using our model with $M = 3$ is approximately 10.4% greater than the revenue under the MNL model. The $(M+1)$-cyclic policy also performs well, with cyclic assortments experiencing only a slight revenue loss compared to the optimal assortments. We also note that the manually decided assortments in practice have low HHI values but carry substantial revenue losses, as in practice, managers may decide the assortments based on many factors that are not observed. These results demonstrate that a wide variety of products with a low frequency can mitigate the satiation effect. Our model achieves this by efficiently balancing products' utility, revenue, and history-dependent effects.

6. Numerical Study

This section presents our numerical studies on synthetic data. Section 6.1 illustrates the efficiency of our formulations regarding computation time and optimality gap compared to other formulations. Section 6.2 shows that our formulation generates significantly more revenue than benchmark heuristics. In Section 6.3, we demonstrate that the bound-free model (12) can scale up to solve large-size instances. Our numerical studies were implemented in the `Julia` (Bezanson et al. 2017), using modeling language `JuMP` (Lubin et al. 2023), on a Virtual Machine with a 32-core Intel Xeon (Skylake, IBRS) @2.39 GHz processor. All MILP models were solved using `Gurobi 11.0.0` (Gurobi Optimization, LLC 2023), and MIECP models were solved using `Mosek 10.2` (MOSEK 2024).

6.1. Formulation Efficiency

We consider four alternate formulations for solving (HAP) without constraints and set the bounds of the no-purchase probability variable as $\rho_L^t = \frac{1}{(1+\sum_{j\in[N]}\exp(\beta_j^0+\sum_{m\in I_j}\beta_j^m))}$ and $\rho_U^t = 1$ for the first three models:

Conic: This is the mixed-integer exponential cone formulation given by (Conic).

Env: This is the mixed-integer linear program obtained by replacing the exponential cone constraint in (Conic) with the convex envelope based constraints (8) and (9), as detailed in Remark 1.

ML: This is the mixed-integer linear formulation based on a multilinear extension of attraction value functions, and its derivation is detailed in the ecompanion EC.5.1.

SCIP: This refers to solving the binary nonlinear model (HAP) by the global optimization solver `SCIP`.

We consider the satiation effects in this numerical study. For a fixed number of products N and memory length M, the base utility of each product β_i^0 is drawn from a uniform $[-1, 1]$ distribution, and its revenue r_i from a uniform $[1, 10]$ distribution. The no-purchase utility is 0. For each $i \in [N]$ and $m \in [M]$, we set the history-dependent effect β_i^m from two uniform distributions $U[-1, 0]$ and $U[-2, -1]$ to represent a *weak* and *strong* satiation effect, denoted as W and S, respectively. We vary $N \in \{30, 50, 70\}$, $M \in \{1, 2\}$, and $T \in \{5, 10\}$. For each combination of N, M, and a satiation level (W or S), we generate five instances.

In EC.6, we provide additional computational results under more general settings. EC.6.1 demonstrates that our model is robust and efficient under both cross-product and cross-period constraints, with performance being more sensitive to the latter. EC.6.2 examines cases with mixed effects and long memory lengths. Our cutting-plane algorithm mentioned in Remark 2 can solve most instances to optimality even when the memory length scales to six.

We solve each instance of (HAP) using one of the formulations Conic, Env, ML, and SCIP. We report the computation time and the end gap, which is defined as $G_{\mathsf{end}} = 100\% \times \frac{R_{\mathsf{U}} - R_{\mathsf{IP}}}{R_{\mathsf{IP}}}$, where R_{U} and R_{IP} is the best upper bound and the best integer solution at termination with a time limit of 3600 seconds and an optimality gap tolerance of 0.5%, respectively.

Table 3 summarizes the computation results. Given a parameter combination, we split 5 instances into two groups: solved and unsolved instances. The solved instances achieved the optimality gap within 3600s, but the unsolved instances failed. #sol indicates the number of solved instances. The average computation time of solved instances is denoted as T_{opt} and the average end gap of unsolved instances as G_{end}. Overall, formulation Env performs best in terms of the number of solved instances, the computation time, and the end gap. Our MILP formulation Env slightly outperforms its conic counterpart, Conic, because the former leverages the convex envelope of attraction value functions. Not surprisingly, our formulations Env and Conic dominate ML and SCIP in all aspects. In particular, our formulations solve more instances to global optimality, with fewer computational times, than ML and SCIP do. For unsolved instances, the end gap of our formulations is significantly smaller than that of ML and SCIP.

Besides the computation time and end gap, it is also helpful to compare the continuous relaxation of different formulations. We compute the root gap by $100\% \times \frac{R_{\mathsf{Rlx}} - R_{\mathsf{IP}}}{R_{\mathsf{IP}}}$, where R_{Rlx} is the objective value of the continuous relaxation of a given formulation. A small root gap indicates a tight formulation. Table 4 reports the average root gap of Conic, Env, and ML under different parameter configurations. Overall, the root gap of our formulations Conic and Env is significantly smaller than that of ML. It indicates the tightness of our

Table 3 Computation time and end gap of four formulations under satiation effects

T	M	N	β	Conic #sol	Conic $T_{\text{opt}(s)}$	Conic $G_{\text{end}(\%)}$	Env #sol	Env $T_{\text{opt}(s)}$	Env $G_{\text{end}(\%)}$	ML #sol	ML $T_{\text{opt}(s)}$	ML $G_{\text{end}(\%)}$	SCIP #sol	SCIP $T_{\text{opt}(s)}$	SCIP $G_{\text{end}(\%)}$
5	1	30	W	5	36	0	5	6	0	5	267	0	0	-	648
5	1	30	S	5	90	0	5	96	0	5	527	0	0	-	920
5	1	50	W	3	65	0.64	5	657	0	3	473	1.04	0	-	1380
5	1	50	S	3	1766	0.66	4	1249	0.68	1	3492	1.51	0	-	1527
5	1	70	W	3	453	1.75	4	102	2.17	3	855	2.26	0	-	1798
5	1	70	S	1	1695	0.87	2	757	0.68	1	2523	1.87	0	-	2191
5	2	30	W	5	267	0	5	95	0	3	1339	0.99	-	-	913
5	2	30	S	3	527	2.39	3	363	1.5	0	-	11.76	0	-	1046
5	2	50	W	3	689	0.95	5	160	0	1	2760	1.57	0	-	1415
5	2	50	S	0	-	2.35	1	2133	1.64	0	-	49.13	0	-	1756
5	2	70	W	0	-	1.81	3	811	1.6	0	-	16.35	0	-	2369
5	2	70	S	1	1020	3.41	1	57	2.37	0	-	79.22	0	-	2459
10	1	30	W	0	-	0.92	4	422	0.86	0	-	0.94	0	-	744
10	1	30	S	0	-	2.14	1	2712	1.77	0	-	2.86	0	-	1005
10	1	50	W	0	-	1.35	2	1994	1.4	0	-	1.48	0	-	1477
10	1	50	S	0	-	3.82	0	-	3.12	0	-	3.88	0	-	1577
10	1	70	W	0	-	2.44	1	313	2.06	0	-	3.27	0	-	1907
10	1	70	S	0	-	3.08	0	-	1.9	0	-	4.43	0	-	2305
10	2	30	W	0	-	2.37	0	-	1.4	0	-	4.15	0	-	1001
10	2	30	S	0	-	7.52	0	-	5.8	0	-	165.07	0	-	1151
10	2	50	W	0	-	2.9	0	-	2	0	-	63.85	0	-	1563
10	2	50	S	0	-	8.73	0	-	6.05	0	-	608.16	0	-	1891
10	2	70	W	0	-	4.66	0	-	3.05	0	-	283.46	0	-	2756
10	2	70	S	0	-	6.51	0	-	4.64	0	-	750.15	0	-	2763

Table 4 Root gap (%) of continuous relaxation of four formulations under satiation effects

T	M	N	β	Conic	Env	ML	SCIP
5	1	30	W	3.86	3.22	156.73	-
5	1	30	S	6.81	5.89	301.08	-
5	1	50	W	2.89	2.5	295.99	-
5	1	50	S	5.64	5.04	517.19	-
5	1	70	W	3.95	3.55	400.53	-
5	1	70	S	3.84	3.45	713.79	-
5	2	30	W	4.57	4.57	224.85	-
5	2	30	S	13.17	13.17	587.57	-
5	2	50	W	4.77	4.77	430.75	-
5	2	50	S	10.1	10.06	921.6	-
5	2	70	W	5.47	5.3	717.98	-
5	2	70	S	6.85	6.43	1243.68	-

T	M	N	β	Conic	Env	ML	SCIP
10	1	30	W	4.38	3.65	179.23	-
10	1	30	S	7.83	6.56	343.77	-
10	1	50	W	3.3	2.82	336.43	-
10	1	50	S	7.11	5.64	588.26	-
10	1	70	W	5.17	4.77	455.25	-
10	1	70	S	5.05	4.2	810.67	-
10	2	30	W	5.64	5.61	275.29	-
10	2	30	S	15.4	15.32	786.57	-
10	2	50	W	5.9	5.76	539.85	-
10	2	50	S	13.34	12.09	1419.53	-
10	2	70	W	6.94	6.46	940.42	-
10	2	70	S	9.02	7.85	1890.09	-

formulations. Moreover, the root gap of our formulations is stable and does not scale with the problem size. For instance, the root gap of Env is no more than 7% when $M = 1$ and no more than 16% when $M = 2$.

6.2. Performance of Heuristic Policies

We compare the revenue generated by our formulation against two heuristic policies without constraints.

Sequential – RO: The sequential RO-policy shown in Algorithm 1.

$\mathsf{Sequential-LOSPO}$: For each period, this policy drops one satiation product that causes the greatest revenue reduction from the base assortment obtained from the $\mathsf{Sequential-RO}$ policy. We refer to it as *sequential leave-one-satiation-product-out* (Sequential-LOSPO) policy.

We consider instances with mixed effects. A product has an addiction (resp. satiation) effect with probability θ (resp. $1-\theta$), where $\theta \in \{0, 0.1, 0.2\}$. When $\theta = 0$, all products have satiation effects. For a product with addiction (resp. satiation) effect, its $\{\beta_i^m\}_{m\in[M]}$ are uniformly sampled from $[0,1]$ (resp. $[-2,-1]$).

We compare revenue obtained by the two heuristic policies and our formulation. We define the relative revenue gap as $100\% \times \frac{R_{\mathsf{Env}} - R_{\mathsf{heuristic}}}{R_{\mathsf{Env}}}$, where $R_{\mathsf{heuristic}}$ is revenue generated by one of two heuristic policies. R_{Env} is computed by Env within 3600 seconds. Table 5 shows that the revenue gap of both heuristics is not negligible, especially when θ is small. For instance, the smallest relative gap is at least more than 10% if $\theta = 0$, suggesting that our formulation has more advantage in scenarios where the satiation effect is dominant. Although the revenue gap shrinks when θ increases, our formulation still holds value as it has an optimality guarantee and is easily extended to constrained cases.

Table 5 Revenue gap (%) of heuristic policies under different history-dependent effects with $M = 2$

		Sequential − RO			Sequential − LOSPO		
N	**T**	$\theta=0$	$\theta=0.1$	$\theta=0.2$	$\theta=0$	$\theta=0.1$	$\theta=0.2$
30	5	18.60	13.02	2.99	15.09	11.77	2.60
30	10	31.37	19.43	4.18	28.47	18.65	3.81
50	5	14.39	5.18	2.02	12.90	4.63	1.84
50	10	25.72	7.73	2.82	24.20	7.27	2.66
70	5	12.31	4.00	3.59	10.91	3.80	3.40
70	10	21.86	5.74	5.03	20.61	5.59	4.88

6.3. Computing $(M+1)$-Cyclic Policies

In this section, we demonstrate the efficiency and scalability of the bound-free model (12) for computing $(M+1)$-cyclic policies under the non-overlapping constraint. Note that due to the presence of $\mathbf{\Gamma}$ variables, the number of variables in (12) is $\mathcal{O}(N^2)$. To address this issue, we propose a projected-cutting-plane algorithm to obtain a projected approximation of (12). The cutting plane algorithm first sets a base formulation without using $\mathbf{\Gamma}$ variables, that is, constraints (12c)-(12e), and solves its continuous relaxation to obtain a continuous solution. Then, the algorithm calls a separation oracle to generate cuts violated by the continuous solution, which are selected from the projection of constraints (12c)-(12e) onto the space of the base formulation, i.e., the space of $(\boldsymbol{x}, \boldsymbol{\rho}, \boldsymbol{\gamma})$ variables. Adding new cuts to the base model leads to a tighter formulation. Running the separation oracle once is a round of cut generation. We can introduce new cuts over K rounds and refer to the resulting formulation as BF-K, where “BF” stands for “bound-free”. We refer to EC.5.2 for a detailed description of the projected cutting-plane algorithm.

Clearly, a larger K leads to a tighter formulation at the cost of adding cuts. To select a proper K that balances the formulation tightness and the formulation size, we compute the optimality gap closed after the K^{th} round as follows:

$$\mathsf{GClosed}_K = 100\% \times \frac{\mathsf{Opt}_0 - \mathsf{Opt}_K}{\mathsf{Opt}_0 - \mathsf{Opt}_\infty},$$

where Opt_K is the optimal objective value of the continuous relaxation of BF-K, and the index 0 (resp. ∞) refers to the base (resp. limiting) model. Table 6a reports the gap closed by BF-K for $K \in \{1,2,3\}$ on instances with $M \in \{1,2\}$ and $N \in \{50,100\}$. BF-1 closes about 80% of the optimality gap, suggesting setting $K = 1$ is good enough.

Table 6 The computational performance of (12) with a projected-cutting-plane implementation

(a) Performance of the cutting-plane algorithm

N	M	$\mathsf{GClosed}_1$	$\mathsf{GClosed}_2$	$\mathsf{GClosed}_3$
50	1	79.25	91.19	96.21
50	2	82.24	93.14	95.90
100	1	89.82	95.61	97.58
100	2	77.24	90.30	93.48

(b) Performance of bound-free model BF-1

N	M	#sol	$T_{\mathsf{opt}(s)}$	$G_{\mathsf{end}(\%)}$
50	1	5	0.57	0.30
50	2	5	70.74	0.46
100	1	5	8.19	0.42
100	2	5	378.59	0.49

Table 6b reports the performance of BF-1. The result demonstrates the scalability and efficiency of the bound-free model compared to (CONIC). BF-1 solves all instances to optimality within 400 seconds. In contrast, under the number of products $N = 50$ and memory length $M = 2$, both Conic and Env fail to solve some of the instances as shown in Table 3.

In EC.6.3, we also evaluate $(M+1)$-cyclic policies, which yield a small revenue gap, particularly when T is large, and consistently outperform both heuristic policies.

7. Conclusion

This paper studies a multi-period assortment planning problem with history-dependent customer utility. We prove that the problem is NP-hard. To address this difficulty, we develop an MIECP formulation to find a global optimal solution. We also prove the optimality of the sequential policy for the addiction case and the asymptotic optimality of cyclic policies while relating the cycle length to customer memory length.

Our research can be extended in several directions. First, it is important to develop efficient approximation algorithms, such as fully polynomial-time approximation schemes, to solve (HAP), especially for cases with large memory lengths and long planning horizons. Second, it is intriguing to establish an inapproximability result for (HAP). Third, extending the model to online settings for personalized assortments or dynamic settings to adjust assortments to quickly respond to stochastic demand. Fourth, it is worthwhile to explore joint assortment, pricing, or inventory decisions under history-dependent effects. The last one is to explore history-dependent effects in other choice models for assortment optimization.

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Additional Discussions, Missing proofs, and Formulations

EC.1. Meituan's Grocery Flash Sale Channel

Figure EC.1 details Meituan's flash sale channel. Figure EC.1a is the main page of the Meituan online store. The main page lists the entry of the grocery flash sale channel and highlights its remaining time in the current campaign. The flash sale channel runs each campaign on a daily basis. Hence, the remaining time in our example is no more than 24 hours. Figure EC.1b is the screenshot of the flash sale's products. In each category, only a limited number of products are offered. Please note that the products are the same for all customers.

Meituan aggregates all orders during a flash sale campaign and forwards the order information to suppliers once the current campaign ends. Suppliers will then prepare the products and deliver them to the pick-up stations the next day. Based on our discussion with Meituan, they must determine the products for flash sales for a particular planning period ahead of time because Meituan needs lead time to negotiate contracts with suppliers. As shown in Figure EC.1c, the payment page details the delivery and pick-up information, such as the customer's chosen pick-up station and the expected pick-up time. This time indicates when the product is expected to reach the pick-up station. If a customer opts for home delivery, the products will initially be sent to the pick-up station, after which the station's staff will deliver them to the customer's address. This case shows that inventory is not the key concern of the platform because all the products are delivered from suppliers to the pick-up station the next day.

To the best of our knowledge, Meituan plans assortments for its flash sales channel independently. This makes our history-dependent assortment planning model well-suited to this relatively clean setting. If a platform also has a regular sales channel, the flash sale campaigns may include products that are simultaneously available outside of the promotion period, potentially influencing the demand of these regular channel products. This leaves an interesting problem of how to plan assortments for both channels under this additional intertemporal effect. In this setting, our model can handle assortment planning for a single channel by incorporating the terms of product availability in the other channel into the utility function.

EC.2. Proofs of Section 2

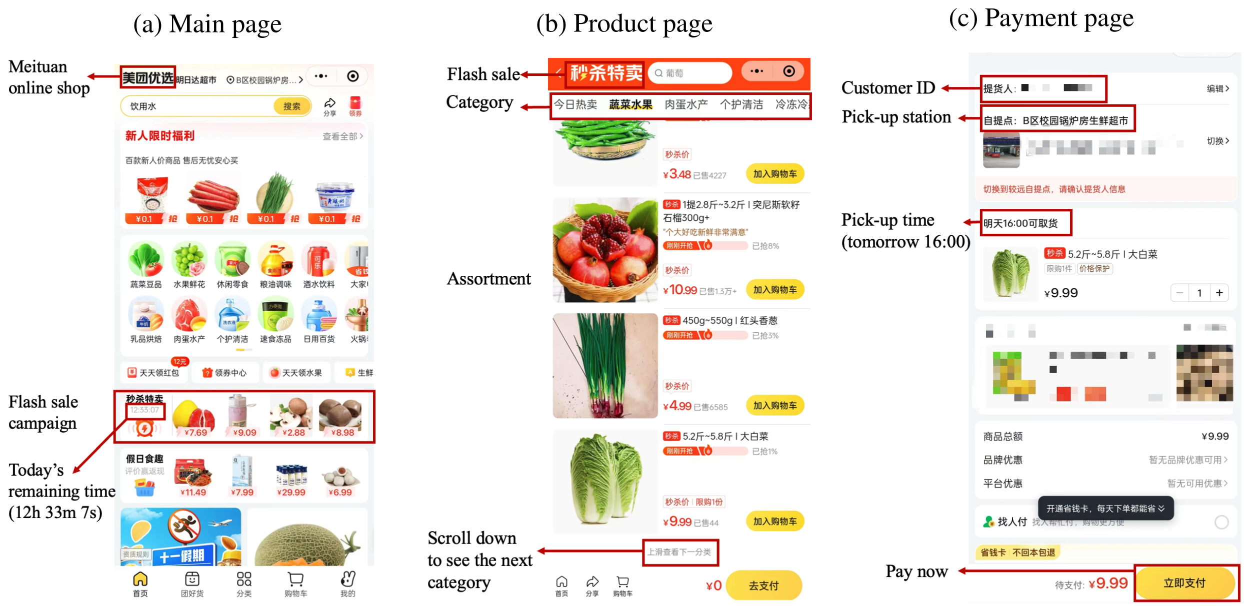
EC.2.1. Proof of Proposition 1

PROPOSITION 1. (HAP) *is NP-hard even when the planning horizon is two, the memory length is one, and history-dependent effects are negative.*

Proof. To prove the NP-hardness of (HAP), we will reduce an NP-hard problem—the *3/4-Partition* problem, to a special case of (HAP). The 3/4-Partition problem is defined as follows:

- **Input:** A set $\{c_1, c_2, \ldots, c_N\}$ of non-negative integers.
- **Output:** True if and only if there exists a subset $S \subseteq [N]$ such that $\sum_{i\in S} c_i = (3/4)\sum_{i\in[N]} c_i$.

Figure EC.1 **Screenshots of Meituan's grocery flash sale channel**



We will show the NP-hardness of the 3/4-Partition problem by showing that any instance of a *Partition* problem, a well-known NP-hard problem, can be reduced to an instance of the 3/4-Partition problem. The Partition problem is defined as follows:

- **Input:** A set $\{w_1, w_2, \ldots, w_N\}$ of non-negative integers.
- **Output:** True if and only if there exists a subset $S \subseteq [N]$ such that $\sum_{i\in S} w_i = \sum_{i\in[N]/S} w_i$.

Notice that $\sum_{i\in S} w_i = \sum_{i\in[N]/S} w_i$ if and only if $\sum_{i\in S} w_i = (1/2)\sum_{i\in[N]} w_i$. Given an instance of Partition problem, we solve it by solving a 3/4-Partition problem as follows. Define $c_i = w_i$ for $i \in [N]$ and $c_{N+1} = \sum_{i\in[N]} w_i$. Then, solve 3/4-Partition with input $(c_1, \ldots, c_{N+1})$. If the output is true and returns an index set S such that $\sum_{i\in S} c_i = \frac{3}{4}\sum_{i\in[N+1]} c_i$ then the output for Partition is true, that is, $\sum_{i\in S\setminus\{N+1\}} w_i = \sum_{i\in[N+1]\setminus S} w_i$ is a partition. The correctness follows from the fact that the set S contains $N+1$. If the output is false then the output for Partition is also false since otherwise adding c_{N+1} into one of two sets yields a 3/4 partition.

Next, we introduce a special case of (HAP). In such a special case, the planning horizon $T = 2$, the memory length $M = 1$, and all products have identical revenue, that is, $r_i = r > 0$ for $i \in [N]$. In addition, we assume that the history-dependent effects are homogeneous, that is, for every $i \in [N]$, $\exp(\beta_i^1) = k$ for some $k \in (0,1)$. Then, it turns out that (HAP) becomes

$$\max_{\boldsymbol{x}^1,\boldsymbol{x}^2\in\{0,1\}^N} \frac{r}{2}\left\{\frac{\sum_{i\in[N]} x_i^1 \exp(\beta_i^0)}{1+\sum_{i\in[N]} x_i^1 \exp(\beta_i^0)} + \frac{\sum_{i\in[N]} x_i^2 \exp\left(\beta_i^0 + x_i^1\beta_i^1\right)}{1+\sum_{i\in[N]} x_i^2 \exp\left(\beta_i^0 + x_i^1\beta_i^1\right)}\right\}. \tag{SPECIAL}$$

We simplify (SPECIAL) by showing that the optimal assortment in the second period is $x_i^2 = 1$ for $i \in [N]$. Since all products have the same revenue, we can focus on maximizing the purchase probability. Given an assortment $\boldsymbol{x}^1 \in \{0,1\}^N$, the purchase probability at the first period is a constant, and the purchase probability at the second period increases with $\sum_{i\in[N]} x_i^2 \exp\big(\beta_i^0 + x_i^1\beta_i^1\big)$. Since $\exp\big(\beta_i^0 + x_i^1\beta_i^1\big) > 0$ for each $i \in [N]$, then the optimal solution at the second period is $x_i^2 = 1$ for $i \in [N]$. Let $\nu_i = \exp(\beta_i^0)$ be the base attraction value and $V = \sum_{i\in[N]} \exp(\beta_i^0) = \sum_{i\in[N]} \nu_i$ denote the total base attraction value. Thus, solving (SPECIAL) equals to solving the following model:

$$\max_{\boldsymbol{x}^1\in\{0,1\}^N} \frac{r}{2}\left\{\frac{\sum_{i\in[N]} x_i^1\nu_i}{1+\sum_{i\in[N]} x_i^1\nu_i} + \frac{V-\sum_{i\in[N]} x_i^1\nu_i + k\sum_{i\in[N]} x_i^1\nu_i}{1+V-\sum_{i\in[N]} x_i^1\nu_i + k\sum_{i\in[N]} x_i^1\nu_i}\right\}. \tag{EC.1}$$

Next, we show that the 3/4-Partition problem can be reduced to our special case (SPECIAL). First, define $\nu_i = ac_i$ for $i \in [N]$, $k = 1/4$, and $C = \sum_{i\in[N]} c_i$, where

$$a = \frac{4(1-\sqrt{1-k})}{C(3(1-k)+3\sqrt{1-k}-4)} = \frac{16-8\sqrt{3}}{C(6\sqrt{3}-7)} > 0.$$

Hence, $V = \sum_{i\in[N]} \nu_i = aC$. Second, set the target average revenue as

$$\frac{r}{2}\Big(\frac{3aC}{4+3aC} + \frac{7aC}{16+7aC}\Big).$$

The Partition problem indeed has a solution if and only if there exists a $\boldsymbol{x}^1 \in \{0,1\}^N$ such that

$$\begin{aligned}
&\frac{r}{2}\left\{\frac{\sum_{i\in[N]} x_i^1\nu_i}{1+\sum_{i\in[N]} x_i^1\nu_i} + \frac{V-\sum_{i\in[N]} x_i^1\nu_i + k\sum_{i\in[N]} x_i^1\nu_i}{1+V-\sum_{i\in[N]} x_i^1\nu_i + k\sum_{i\in[N]} x_i^1\nu_i}\right\}\\
=&\frac{r}{2}\left\{\frac{a\sum_{i\in[N]} x_i^1 c_i}{1+a\sum_{i\in[N]} x_i^1 c_i} + \frac{aC-(1-k)a\sum_{i\in[N]} x_i^1 c_i}{1+aC-(1-k)a\sum_{i\in[N]} x_i^1 c_i}\right\}\\
\le&\max_{y\in[0,C]} \frac{r}{2}\left\{\frac{ay}{1+ay} + \frac{aC-(1-k)ay}{1+aC-(1-k)ay}\right\} = \frac{r}{2}\Big(\frac{3aC}{4+3aC} + \frac{7aC}{16+7aC}\Big).
\end{aligned}$$

This holds because if we view the objective function in the last row as a one-dimension function, denoted as

$$h(y) := \frac{r}{2}\left(\frac{ay}{1+ay} + \frac{aC-(1-k)ay}{1+aC-(1-k)ay}\right),$$

$h(y)$ is concave in y and achieves a unique maximum at $y = 3C/4$. □

EC.3. Proofs in Section 3.1

EC.3.1. Proof of Proposition 2

To prove this proposition, we will invoke the following technical lemma.

LEMMA EC.1. *Let $f : \{0,1\}^n \to \mathbb{R}$ and $L : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible affine transformation such that $L(\{0,1\}^n) = \{0,1\}^n$, and consider a composition $(f\circ L)(\boldsymbol{x}) := f(L(\boldsymbol{x}))$ for every $\boldsymbol{x} \in \{0,1\}^n$. Then,*

$$\mathrm{conc}(f)(\boldsymbol{x}) = \mathrm{conc}(f\circ L)\big(L^{-1}(\boldsymbol{x})\big) \quad \text{for } \boldsymbol{x} \in [0,1]^n.$$

Proof. Let $g(\boldsymbol{x}) := \operatorname{conc}(f\circ L)\big(L^{-1}(\boldsymbol{x})\big)$ for $\boldsymbol{x}\in[0,1]^n$. Clearly, $g(\boldsymbol{x}) = f(\boldsymbol{x})$ for every $\boldsymbol{x}\in\{0,1\}^n$, and $g(\cdot)$ is a concave function. Thus, we show that $g(\boldsymbol{x}) \geq \operatorname{conc}(f)(\boldsymbol{x})$ for every $\boldsymbol{x}\in[0,1]^n$ since the concave envelope is the smallest concave function on $[0,1]^n$ that overestimates $f(\cdot)$ on $\{0,1\}^n$.

To show the opposite direction, we need the dual definition of the concave envelope Rockafellar (1970), that is,

$$\operatorname{conc}(f)(\boldsymbol{x}) = \max\Bigg\{\sum_{v\in V} f(v)\lambda_v \;\Bigg|\; \boldsymbol{x} = \sum_{v\in V} v\lambda_v,\ \sum_{v\in V}\lambda_v = 1,\ \lambda_v\geq 0 \text{ for } v\in V\Bigg\},$$

where $V := \{0,1\}^n$. Now, consider a point $\bar{\boldsymbol{x}}\in[0,1]^n$ and let $\bar{\boldsymbol{y}} = L^{-1}(\bar{\boldsymbol{x}})$. We will argue that $\operatorname{conc}(f\circ L)(\bar{\boldsymbol{y}}) \leq \operatorname{conc}(f)(\bar{\boldsymbol{x}})$. By the dual definition of concave envelope, there exists a convex multiplier $\{\bar\lambda_v\}_{v\in V}$ such that $\bar{\boldsymbol{y}} = \sum_{v\in V} v\bar\lambda_v$ and $\operatorname{conc}(f\circ L)(\bar{\boldsymbol{y}}) = \sum_{v\in V}(f\circ L)(v)\bar\lambda_v$. It follows readily that $\bar{\boldsymbol{x}} = L(\bar{\boldsymbol{y}}) = \sum_{v\in V} L(v)\bar\lambda_v$, and thus

$$\operatorname{conc}(f)(\bar{\boldsymbol{x}}) \geq \sum_{v\in V} f\big(L(v)\big)\cdot\bar\lambda_v = \sum_{v\in V}(f\circ L)(v)\cdot\bar\lambda_v = \operatorname{conc}(f\circ L)(\bar{\boldsymbol{y}}),$$

where the first inequality holds due to the dual definition of $\operatorname{conc}(f)(\cdot)$ and $\bar{\boldsymbol{x}}$ can be expressed as $\sum_{v\in V} L(v)\bar\lambda_v$. □

PROPOSITION 2 *Constraint (6) is equivalent to the following system of linear inequalities.*

$$y_i^t \leq \alpha_i(\boldsymbol{h}_{i,0}^\sigma)(\gamma_i^t - \tilde z_{i\sigma(1)}^t) + \sum_{k\in[M]}\alpha_i(\boldsymbol{h}_{i,k}^\sigma)(\tilde z_{i\sigma(k)}^t - \tilde z_{i\sigma(k+1)}^t) \qquad \textit{for } \sigma\in\Omega$$

where $\tilde z_{i\sigma(M+1)}^t = 0$, $\tilde z_{i\sigma(k)}^t = z_{i\sigma(k)}^t$ *if* $\sigma(k)\notin I_i$, *and* $\tilde z_{i\sigma(k)}^t = \gamma_i^t - z_{i\sigma(k)}^t$ *if* $\sigma(k)\in I_i$.

Proof. We start with the satiation case. Since $\boldsymbol{\beta}_i$ are non-positive, by Corollary 3.14 in Tawarmalani et al. (2013), $\alpha_i(\cdot)$ is supermodular over $(x_i^{t-1},\dots,x_i^{t-M})$. Then, it follows from Theorem 3.3 in Tawarmalani et al. (2013) that the concave envelope of $\alpha_i(\cdot)$ is given as:

$$\operatorname{conc}(\alpha_i)(x_i^{t-1},\dots,x_i^{t-M}) = \min_{\sigma\in\Omega}\Bigg\{\alpha_i(\boldsymbol{w}_0^\sigma)(1-x_i^{t-\sigma(1)}) + \sum_{k\in[M]}\alpha_i(\boldsymbol{w}_k^\sigma)(x_i^{t-\sigma(k)} - x_i^{t-\sigma(k+1)})\Bigg\}, \tag{EC.2}$$

where $\boldsymbol{w}_0^\sigma = \boldsymbol{0}$, $\boldsymbol{w}_k^\sigma = \boldsymbol{w}_{k-1}^\sigma + \boldsymbol{e}_{\sigma(k)}$ for $k\in[M]$, and $x_i^{t-\sigma(M+1)} = 0$. It follows readily that its perspective function can be represented as follows under the presence of constraint (4):

$$\operatorname{pers}(\operatorname{conc}(\alpha_i))(\gamma_i^t,\boldsymbol{z}_i^t) = \min_{\sigma\in\Omega}\Bigg\{\alpha_i(\boldsymbol{w}_0^\sigma)(\gamma_i^t - z_{i\sigma(1)}^t) + \sum_{k\in[M]}\alpha_i(\boldsymbol{w}_k^\sigma)(z_{i\sigma(k)}^t - z_{i\sigma(k+1)}^t)\Bigg\},$$

where $z_{i\sigma(M+1)}^t = 0$

For the general case of mixed satiation and addiction effects, we transform the variable x_i^{t-m} to $1-x_i^{t-m}$ if $\beta_i^m > 0$, and obtain a transformed attraction value function, defined as follows:

$$\begin{aligned}\bar\alpha_i(w^1,\dots,w^M) &:= \exp\Bigg(\beta_i^0 + \sum_{k\in I_i}\beta_i^k(1-w^k) + \sum_{k\in[M]\setminus I_i}\beta_i^k w^k\Bigg)\\ &= \exp\Bigg(\beta_i^0 + \sum_{k\in I_i}\beta_i^k + \sum_{k\in[M]}(-|\beta_i^k|)w^k\Bigg).\end{aligned}$$

Now, the coefficients $-|\beta_i^k|$ are non-positive, and by Corollary 3.14 in Tawarmalani et al. (2013), the transformed function $\bar{\alpha}_i(\cdot)$ is supermodular. Thus, its concave envelope can be described using (EC.2).

Next, we utilize $\text{conc}(\bar{\alpha}_i)(\cdot)$ to characterize the concave envelope of the original attraction value function $\alpha_i(\cdot)$. Define $\tilde{x}_i^{t-m} = x_i^{t-m}$ if $m \in [M] \setminus I_i$, $\tilde{x}_i^{t-m} = 1 - x_i^{t-m}$ if $m \in I_i$, and $\tilde{x}_i^{t-\sigma(M+1)} = 0$. We then build the following connections:

$$\begin{aligned}
\text{conc}(\alpha_i)(x_i^{t-1}, \ldots, x_i^{t-M}) &= \text{conc}(\bar{\alpha}_i)(\tilde{x}_i^{t-1}, \ldots, \tilde{x}_i^{t-M}) \\
&= \min_{\sigma \in \Omega} \Big\{ \bar{\alpha}_i(\boldsymbol{w}_0^\sigma)(1 - \tilde{x}_i^{t-\sigma(1)}) + \sum_{k \in [M]} \bar{\alpha}_i(\boldsymbol{w}_k^\sigma)(\tilde{x}_i^{t-\sigma(k)} - \tilde{x}_i^{t-\sigma(k+1)}) \Big\} \\
&= \min_{\sigma \in \Omega} \Big\{ \alpha_i(\boldsymbol{h}_{i,0}^\sigma)(1 - \tilde{x}_i^{t-\sigma(1)}) + \sum_{k \in [M]} \alpha_i(\boldsymbol{h}_{i,k}^\sigma)(\tilde{x}_i^{t-\sigma(k)} - \tilde{x}_i^{t-\sigma(k+1)}) \Big\}
\end{aligned}$$

The first equality follows from Lemma EC.1, the second equality holds by (EC.2), and the last equality holds since it follows from the definition of $\boldsymbol{h}$ in (SWITCHNESTED) that $\bar{\alpha}_i(\boldsymbol{w}_k^\sigma) = \alpha_i(\boldsymbol{h}_{ik}^\sigma)$.

Finally, we scale the concave envelope to obtain

$$\text{pers}(\text{conc}(\alpha_i))(\gamma_i^t, \boldsymbol{z}_i^t) = \min_{\sigma \in \Omega} \Big\{ \alpha_i(\boldsymbol{h}_{i,0}^\sigma)(\gamma_i^t - \tilde{z}_{i\sigma(1)}^t) + \sum_{k \in [M]} \alpha_i(\boldsymbol{h}_{i,k}^\sigma)(\tilde{z}_{i\sigma(k)}^t - \tilde{z}_{i\sigma(k+1)}^t) \Big\}.$$

This completes the proof. □

EC.3.2. Proof of Theorem 1

THEOREM 1 (CONIC) *is a mixed-integer exponential cone formulation of problem* (HAP).

Proof. By lifting, we equivalently decompose (CHOICE) into the choice constraints (2):

$$\begin{aligned}
&\gamma_i^t = \rho^t x_i^t && (2a) \\
&z_{im}^t = \gamma_i^t x_i^{t-m} \quad \text{for } m \in [M] && (2b) \\
&y_i^t = \text{pers}(\alpha_i)(\gamma_i^t, \boldsymbol{z}_i^t) && (2c).
\end{aligned}$$

Thus, proving the equivalence of (CONIC) and (HAP) suffices to show that constraints (3) (4) (5) and (7) construct an equivalent representation of the choice constraints (2). Clearly, since $\boldsymbol{x}$ is binary, (3) (resp. (4)) models constraint (2a) (resp. (2b)). Next, we show constraints (5) and (7) provide an exact representation of (2c). By $\boldsymbol{x} \in \{0,1\}^{N \times T}$ and constraints (3) and (4), constraint (5) equals to

$$y_i^t \geq \rho^t x_i^t \exp(\beta_i^0 + \sum_{m \in [M]} \beta_i^m x_i^{t-m}). \tag{EC.3}$$

On the other hand, the constraint (7) is the concave envelope of $\alpha_i(\cdot)$ scaled by γ_i^t. That is, $y_i^t \leq \gamma_i^t \cdot \text{conc}(\alpha_i)(x_i^{t-1}, \ldots, x_i^{t-M})$. In our case, the concave envelope α_i is the tightest concave extension α_i (see Tawarmalani and Sahinidis 2002, Theorem 6). Thus, $\text{conc}(\alpha_i)(x_i^{t-1}, \ldots, x_i^{t-M}) = \alpha_i(x_i^{t-1}, \ldots, x_i^{t-M})$ for

$x_i^{t-m} \in \{0,1\}$ and $m \in [M]$. Therefore, constraint (7) equals to $y_i^t \le \gamma_i^t \cdot \alpha_i(x_i^{t-1}, \dots, x_i^{t-M})$. Combined with constraint (3), constraint (7) equals to

$$y_i^t \le \rho^t x_i^t \exp(\beta_i^0 + \sum_{m\in[M]} \beta_i^m x_i^{t-m}). \tag{EC.4}$$

Constraints (EC.3) and (EC.4) suggest that constraints (5) and (7) are exact representation of (2c).

Recall that exponential cones are three-dimensional convex cones:

$$\mathcal{K}_{\exp} = \big\{(x_1,x_2,x_3) \mid x_1 \ge x_2 \exp(x_3/x_2), x_2 > 0\big\} \cup \big\{(x_1,0,x_3) \mid x_1 \ge 0, x_3 \le 0\big\}.$$

By introducing new variable $w_i^t = \beta_i^0 \gamma_i^t + \sum_{m\in[M]} \beta_i^m z_{im}^t$, (5) can be represented by exponential cones:

$$(y_i^t, \gamma_i^t, w_i^t) \in \mathcal{K}_{\exp}.$$

Therefore, (CONIC) is an equivalent mixed-integer exponentical cone programming of (HAP).

EC.3.3. Proof of Theorem 2

THEOREM 2 *Assume the absence of cross-product and cross-period constraints. Then, the sequential-revenue-ordered policy solves* (HAP) *if the history-dependent effects are non-negative, that is,* $\boldsymbol{\beta} \ge \mathbf{0}$.

Before proving the result, we present some preliminary results that will be used in our proof. Given a vector of attraction values $\boldsymbol{\nu} \in \mathbb{R}_+^N$, consider a single-period assortment optimization problem under the MNL model given as follows:

$$\max_S \Big\{ \mathrm{MNL}(S,\boldsymbol{\nu}) := \sum_{i\in S} r_i \nu_i / \big(1 + \sum_{i\in S} \nu_i\big) \;\Big|\; S \subseteq [N] \Big\}. \tag{EC.5}$$

We assume that products are indexed such that $r_1 \ge \cdots \ge r_N$. For $\kappa \in [N]$, we will use $[\kappa]$ to denote a revenue-ordered (RO) assortment. Talluri and Van Ryzin (2004) show the optimal revenue is achieved by a RO assortment $[\kappa]$ for some cut-off product $\kappa \in [N]$. Without loss of generality, we assume that there exists a unique optimal solution; otherwise, we will use the optimal RO assortment with the largest number of products.

LEMMA EC.2. *Let* $\boldsymbol{\nu} \in \mathbb{R}_+^N$ *and assume that* $r_1 \ge \cdots \ge r_N$. *Then, we obtain the following results:*

1. *The RO assortment* $[\kappa]$ *is optimal if and only if* $r_\kappa \ge \mathrm{MNL}([\kappa],\boldsymbol{\nu})$ *and* $r_{\kappa+1} < \mathrm{MNL}([\kappa],\boldsymbol{\nu})$.
2. *If the RO assortment* $[\kappa]$ *is optimal then, for every* $\boldsymbol{\mu}$ *such that* $\boldsymbol{\mu} \ge \boldsymbol{\nu}$, $\mathrm{MNL}([\kappa],\boldsymbol{\mu}) \ge \mathrm{MNL}([\kappa],\boldsymbol{\nu})$.

Proof of Lemma EC.2. Part 1 follows from (Talluri and Van Ryzin 2004). To prove Part 2, we consider $\boldsymbol{\mu}$ such that $\mu_j \ge \nu_j$ for some $j \in [N]$ but $\mu_i = \nu_i$ for all $i \ne j$, and observe that

$$\mathrm{MNL}([\kappa],\boldsymbol{\mu}) - \mathrm{MNL}([\kappa],\boldsymbol{\nu}) = \begin{cases} \frac{\mu_j - \nu_j}{1+\sum_{i\in[\kappa]}\mu_i}\big(r_j - \mathrm{MNL}([\kappa],\boldsymbol{\nu})\big) & \text{if } j \in [\kappa] \\ 0 & \text{otherwise.} \end{cases}$$

Due to Part 1, we obtain that $r_j \ge \mathrm{MNL}([\kappa],\boldsymbol{\nu})$ for $j \in [\kappa]$. This completes the proof. □

The next result establishes that assortments generated by our policy are nested.

LEMMA EC.3. *Let $[\kappa^1], \ldots, [\kappa^T]$ be a sequence of RO assortments generated by the sequential-revenue-ordered policy. Then, $\kappa^1 \geq \kappa^2 \cdots \geq \kappa^T$.*

Proof of Lemma EC.3. Let $\boldsymbol{\nu} = (\boldsymbol{\nu}^1, \ldots, \boldsymbol{\nu}^T)$ be corresponding attraction values, that is,

$$\nu_i^t := \alpha_i\big(\mathbb{1}(i \in [\kappa^{t-1}]), \ldots, \mathbb{1}(i \in [\kappa^{t-M}])\big) \qquad \text{for } t \in [T] \text{ and } i \in [N].$$

We prove that $N \geq \kappa^1 \geq \cdots \geq \kappa^t \geq 1$ holds for every $t \in [T]$. Clearly, it holds for $\tau = 1$. Assume that it holds for some $1 \leq \tau < T$. Now, we observe that

$$r_{\kappa^\tau+1} < \mathrm{MNL}([\kappa^\tau], \boldsymbol{\nu}^\tau) \leq \mathrm{MNL}([\kappa^\tau], \boldsymbol{\nu}^{\tau+1}) \leq \mathrm{MNL}([\kappa^{\tau+1}], \boldsymbol{\nu}^{\tau+1}),$$

where the first inequality follows from Part 1 of Lemma EC.2, the second inequality holds due to Part 2 of Lemma EC.2 since $\boldsymbol{\beta} \geq 0$ and $\kappa^1 \geq \cdots \geq \kappa^\tau$ imply that $\boldsymbol{\nu}^{\tau+1} \geq \boldsymbol{\nu}^\tau$, and the last inequality follows from the optimality of the RO assortment $[\kappa^{\tau+1}]$. Thus, $r_{\kappa^\tau+1} < \mathrm{MNL}([\kappa^{\tau+1}], \boldsymbol{\nu}^{\tau+1})$. Therefore, by Part 1 of Lemma EC.2, we can conclude that $\kappa^{\tau+1} \leq \kappa^\tau$. □

Now, we are ready to prove Theorem 2.

Proof. Let Z_* and Z_{RO} be the revenue generated from solving (HAP) and from using the sequential revenue-ordered policy. Clearly, our policy generates a feasible plan and thus $Z_{\mathsf{RO}} \leq Z_*$. To show the reverse, we consider the revenue generated from solving the following optimization problem,

$$Z_{\mathsf{U}} := \frac{1}{T}\sum_{t\in[T]} \max_{S^t}\left\{\mathrm{MNL}(S^t, \boldsymbol{\nu}_{\mathsf{U}}^t) \;\middle|\; S^t \subseteq [N]\right\},$$

where $\nu_{\mathsf{U},i}^t$ is the attraction value of product i in period t obtained by assuming that the product i is offered in all past periods, that is,

$$\nu_{\mathsf{U},i}^t := \alpha_i\big(\mathbb{1}(t-1>0), \ldots, \mathbb{1}(t-M>0)\big).$$

Moreover, for each $t \in [T]$, let $[\kappa_{\mathsf{U}}^t]$ denote the RO assortment that maximizes $\mathrm{MNL}(\cdot, \boldsymbol{\nu}_{\mathsf{U}}^t)$. To complete the proof, we will show that $Z_* \leq Z_{\mathsf{U}} \leq Z_{\mathsf{RO}}$.

Now, we show that $Z_* \leq Z_{\mathsf{U}}$. Let $S_*^1, \ldots, S_*^T$ be an optimal solution of (HAP). Let $\boldsymbol{\nu}_* = (\boldsymbol{\nu}_*^1, \ldots \boldsymbol{\nu}_*^T)$ denote the corresponding attraction values, that is, for $t \in [T]$ and $i \in [N]$,

$$\nu_{*,i}^t := \alpha_i\big(\mathbb{1}(i \in S_*^{t-1}), \ldots, \mathbb{1}(i \in S_*^{t-M})\big),$$

where $S^s := \emptyset$ for $s \leq 0$. Then,

$$Z_* = \frac{1}{T}\sum_{t\in[T]} \mathrm{MNL}(S_*^t, \nu_*^t).$$

Now, for $t \in [T]$, let $\kappa_*^t \in [N]$ such that the RO assortment $[\kappa_*^t]$ maximizes $\mathrm{MNL}(\cdot, \boldsymbol{\nu}_*^t)$. Then, $Z_* \leq Z_{\mathsf{U}}$ follows by observing that

$$\mathrm{MNL}(S_*^t, \boldsymbol{\nu}_*^t) \leq \mathrm{MNL}\big([\kappa_*^t], \boldsymbol{\nu}_*^t\big) \leq \mathrm{MNL}\big([\kappa_*^t], \boldsymbol{\nu}_{\mathsf{U}}^t\big) \leq \mathrm{MNL}\big([\kappa_{\mathsf{U}}^t], \boldsymbol{\nu}_{\mathsf{U}}^t\big),$$

where the first (resp. last) inequality holds since the right-hand-side is the optimal revenue that can be achieved under the attraction value $\boldsymbol{\nu}_*^t$ (resp. $\boldsymbol{\nu}_{\mathsf{U}}^t$), and the second inequality follows from Part 2 of Lemma EC.2 since $\boldsymbol{\beta} \geq 0$ implies $\boldsymbol{\nu}_{\mathsf{U}}^t \geq \boldsymbol{\nu}_*^t$.

Last, we show that $Z_{\mathsf{U}} = Z_{\mathsf{RO}}$. Let $[\kappa_{\mathsf{RO}}^1], \ldots, [\kappa_{\mathsf{RO}}^T]$ be the sequence of RO assortments given by our policy, and let $\boldsymbol{\nu}_{\mathsf{RO}}^t$ be the corresponding attraction value, that is,

$$\nu_{\mathsf{RO},i}^t := \alpha_i\big(\mathbb{1}(i \in [\kappa_{\mathsf{RO}}^{t-1}]), \ldots, \mathbb{1}(i \in [\kappa_{\mathsf{RO}}^{t-M}])\big),$$

where $[\kappa_{\mathsf{RO}}^s] := \emptyset$ for $s \leq 0$. In general, $\boldsymbol{\beta} \geq 0$ implies that $\boldsymbol{\nu}_{\mathsf{RO}}^t \leq \boldsymbol{\nu}_{\mathsf{U}}^t$. However, due to the nested structure on $\kappa_{\mathsf{RO}}^1, \ldots, \kappa_{\mathsf{RO}}^T$ in Lemma EC.3, for each $t \in [T]$, we obtain that $\nu_{\mathsf{RO},k}^t = \nu_{\mathsf{U},k}^t$ for $k \in [\kappa_{\mathsf{RO}}^t]$ and, thus

$$\mathrm{MNL}([k], \boldsymbol{\nu}_{\mathsf{RO}}^t) = \mathrm{MNL}([k], \boldsymbol{\nu}_{\mathsf{U}}^t) \qquad \text{for } k \in [\kappa_{\mathsf{RO}}^t]. \tag{EC.6}$$

Moreover, since $[\kappa_{\mathsf{RO}}^t]$ maximizes $\mathrm{MNL}(\cdot, \boldsymbol{\nu}_{\mathsf{RO}}^t)$, Part 1 in Lemma EC.2 shows that

$$r_{\kappa_{\mathsf{RO}}^t} \geq \mathrm{MNL}([\kappa_{\mathsf{RO}}^t], \boldsymbol{\nu}_{\mathsf{RO}}^t) > r_{\kappa_{\mathsf{RO}}^t + 1}. \tag{EC.7}$$

By (EC.6) and (EC.7), we obtain $r_{\kappa_{\mathsf{RO}}^t} \geq \mathrm{MNL}([\kappa_{\mathsf{RO}}^t], \boldsymbol{\nu}_{\mathsf{U}}^t) > r_{\kappa_{\mathsf{RO}}^t + 1}$. By invoking Part 1 in Lemma EC.2, we obtain that $\kappa_{\mathsf{U}}^t = \kappa_{\mathsf{RO}}^t$. Therefore, we can conclude that

$$\mathrm{MNL}([\kappa_{\mathsf{RO}}^t], \boldsymbol{\nu}_{\mathsf{RO}}^t) = \mathrm{MNL}([\kappa_{\mathsf{RO}}^t], \boldsymbol{\nu}_{\mathsf{U}}^t) = \mathrm{MNL}([\kappa_{\mathsf{U}}^t], \boldsymbol{\nu}_{\mathsf{U}}^t),$$

showing that $Z_{\mathsf{U}} = Z_{\mathsf{RO}}$. □

EC.3.4. Proof of Proposition 3

PROPOSITION 3 *Assume the absence of cross-product and cross-period constraints. Then, there exists an optimal assortment $\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^T)$ of (HAP) such that $\big\{i \in [N] \,\big|\, \sum_{t \in T} x_i^t \geq 1\big\}$ is revenue-ordered.*

Proof. Let $\boldsymbol{x} = (\boldsymbol{x}^1, \ldots, \boldsymbol{x}^T)$ be an optimal solution, and let $\boldsymbol{\nu}^t = (\nu_i^t)_{i \in [N]}$ be the attraction value vector in period t, that is, $\nu_i^t = \alpha_i(x_i^{t-1}, \ldots, x_i^{t-M})$. For each $t \in [T]$, let S^t denote $\{i \in [N] \mid x_i^t = 1\}$, and define $S := \cup_{t \in [T]} S^t$. Suppose that S is not revenue-ordered. We assume that products are indexed such that $r_1 \geq \cdots \geq r_N$, then there exists a pair of products, i and j with $i < j$, such that $i \notin S$ but $j \in S$. We consider two possible cases on the revenue of the i^{th} product.

Suppose that there exists $\tau \in [T]$ such that $\mathrm{MNL}(S^\tau, \boldsymbol{\nu}^\tau) \leq r_i$, where $\mathrm{MNL}(\cdot, \cdot)$ is defined as in (EC.5). Let $\mathrm{Rev}_{\mathsf{HAP}}(Z^1, \ldots, Z^T)$ be the objective value of (HAP) if Z^t is the assortment for the t^{th} period. It follows readily that

$$\begin{aligned}
\mathrm{Rev}_{\mathsf{HAP}}(S^1, \ldots, S^{\tau-1}, S^\tau \cup \{i\}, S^{\tau+1}, \ldots, S^T) &= \frac{1}{T}\left(\sum_{t \in [T] \setminus \{\tau\}} \mathrm{MNL}(S^t, \boldsymbol{\nu}^t) + \mathrm{MNL}(S^\tau \cup \{i\}, \boldsymbol{\nu}^\tau)\right) \\
&\geq \frac{1}{T} \sum_{t \in [T]} \mathrm{MNL}(S^t, \boldsymbol{\nu}^t) = \mathrm{Rev}_{\mathsf{HAP}}(S^1, \ldots, S^T),
\end{aligned}$$

where the first equality holds since $i \notin S^t$ for all $t \in [T]$. The inequality holds because $\mathrm{MNL}(S^\tau \cup \{i\}, \boldsymbol{\nu}^\tau) = \lambda r_i + (1-\lambda)\,\mathrm{MNL}(S^\tau, \boldsymbol{\nu}^\tau) \geq \mathrm{MNL}(S^\tau, \boldsymbol{\nu}^\tau)$, where $\lambda = \nu_i^\tau / (1 + \sum_{k \in S^\tau} \nu_k^\tau + \nu_i^\tau)$. If $\mathrm{MNL}(S^\tau, \boldsymbol{\nu}^\tau) = r_i$, then we obtain a new optimal solution with the same revenue as that of $S^1, \ldots, S^T$. If $\mathrm{MNL}(S^\tau, \boldsymbol{\nu}^\tau) < r_i$, the condition is not possible because the above inequality contradicts with the optimality of $S^1, \ldots, S^T$.

Now, assume that $\mathrm{MNL}(S^t, \boldsymbol{v}^t) > r_i$ for all $t \in [T]$. However, this is not possible as we argue next. Let τ be the last period such that $j \in S^\tau$. Then, we have $\mathrm{MNL}(S^t, \boldsymbol{\nu}^t) > r_j$ for all $t \in [T]$. We obtain a contradiction to the optimality of $S^1, \ldots, S^T$ as follows.

$$\begin{aligned}\mathrm{Rev}_{\mathsf{HAP}}(S^1, \ldots, S^{\tau-1}, S^\tau \setminus \{j\}, S^{\tau+1}, \ldots, S^T) &= \frac{1}{T}\left(\sum_{t \in [T]\setminus\{\tau\}} \mathrm{MNL}(S^t, \boldsymbol{\nu}^t) + \mathrm{MNL}(S^\tau \setminus \{j\}, \boldsymbol{\nu}^\tau)\right) \\ &> \frac{1}{T}\sum_{t \in [T]} \mathrm{MNL}(S^t, \boldsymbol{\nu}^t) = \mathrm{Rev}_{\mathsf{HAP}}(S^1, \ldots, S^T),\end{aligned}$$

where the first equality holds due to $j \notin S^t$ for $t \geq \tau + 1$. The inequality holds because $\mathrm{MNL}(S^\tau, \boldsymbol{\nu}^\tau) = \lambda r_j + (1-\lambda)\,\mathrm{MNL}(S^\tau \setminus \{j\}, \boldsymbol{\nu}^\tau) < \lambda\,\mathrm{MNL}(S^\tau, \boldsymbol{\nu}^\tau) + (1-\lambda)\,\mathrm{MNL}(S^\tau \setminus \{j\}, \boldsymbol{\nu}^\tau)$, where $\lambda = \nu_j^\tau / (1 + \sum_{k \in S^\tau} \nu_k^\tau)$. The above inequality contradicts with the optimality of $S^1, \ldots, S^T$. $\square$

EC.4. Proofs of Section 4

We derive a graph representation of (HAP) for proving cyclic policies. With each instance of (HAP), we associate a directed graph $D = (V, A)$ with nodes V and arcs A, referred to as *assortment graph*, as follows. A node in the assortment graph is a tuple of historical assortments, $(\boldsymbol{x}^{t-1}, \ldots, \boldsymbol{x}^{t-M})$. That is, a node $v = (v^1, \ldots, v^M)$ in V has M elements, and each one is an assortment. The first element denotes the latest offered assortment, and the last one is the earliest one in memory. The number of nodes is 2^{NM}. We say a node $\mu = (\mu^1, \ldots, \mu^M)$ is a predecessor of $v = (v^1, \ldots, v^M)$ if $\mu^k = v^{k+1}$ for $k = 1, 2, \ldots, M-1$. Then, an ordered pair (μ, v) is an arc going from μ to v if μ is a predecessor of v. Therefore, the arc set in the assortment graph is given as follows:

$$A := \{(\mu, v) \mid \text{ if } \mu \text{ is predecessor of } v \text{ and } \mu, v \in V\}.$$

The number of arcs is 2^{NM+N}. The arc (μ, v) records the latest assortment v^1 and drops the earliest one μ^M, which defines the updating process of history assortments within memory. Then, we can define an arc's weight as

$$w(\mu, v) = \sum_{i \in [N]} r_i \pi_i(v^1, \mu^1, \ldots, \mu^M)$$

where $\pi_i(\cdot)$ is the purchase probability of i given current assortment v^1 and historical assortments $(\mu^1, \ldots, \mu^M)$ defined in Section 2.1. Here, we omit the superscript of period t for simplicity.

EC.4.1. Proof of Theorem 3

THEOREM 3 *Given an instance of* (INFTY)*, there exists a positive integer L^* such that L^*-cyclic policy is optimal to* (INFTY).

Proof. We show that finding an optimal assortment planning for (INFTY) equals finding the maximum mean cycle in the assortment graph, which is the cycle with the largest mean weight.

First, we show an assortment planning of (INFTY) maps to a path in the assortment graph. Suppose $\boldsymbol{x}^1, \boldsymbol{x}^2, \dots$ is an assortment planning of (INFTY), it defines a path in the assortment graph, that is, $v_{\mathbf{0}}, v(1), v(2), \dots$, where $v_{\mathbf{0}} = (\mathbf{0}, \dots, \mathbf{0})$ and

$$v(t)^k = \begin{cases} \boldsymbol{x}^{t-k+1} & \text{if } t-k \geq 0 \\ \mathbf{0} & \text{otherwise,} \end{cases} \qquad \text{for } k \in [M] \text{ and } t \in \mathbb{Z}_+ .$$

It suggests $v(t)$'s first element records the assortment decision in period t, and the following $M-1$ elements record the historical assortments before $\boldsymbol{x}^t$. Then, the mean weight of the path equals to the average revenue of the assortment planning. That is,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{t\in[T]} w\big(v(t-1), v(t)\big) = \lim_{T\to\infty} \frac{1}{T} \sum_{t\in[T]} \sum_{i\in[N]} r_i \pi_i^t(\boldsymbol{x}^t, \boldsymbol{x}^{t-1} \dots, \boldsymbol{x}^{t-M}).$$

Next, we show that to solve (INFTY), it suffices to find the maximum mean cycle denoted as C^*. Since any assortment planning is a path in the assortment graph, solving (INFTY) equals finding an infinite-length path with the largest mean weight and the starting node $v_{\mathbf{0}}$. We call such a path an optimal path. Because the assortment graph has finite nodes and arcs, an optimal path will inevitably repeat cycles in the assortment graph. Given an infinite-length path, we can repeatedly replace each cycle in the path with the maximum mean cycle. Finally, we obtain a path that repeats the maximum mean cycle forever and has a larger mean weight than the initial one. We do not need to consider the path's weight from the starting node $v_{\mathbf{0}}$ to the maximum mean cycle because it will vanish when taking the average over an infinite horizon T.

Next, we show how to recover an assortment planning from the maximum mean cycle. Suppose the maximum mean cycle is $C^* = v(1), v(2), \dots, v(L^*-1), v(0)$ where L^* is the length of the maximum mean cycle. We can construct a cyclic policy as follows.

$$\boldsymbol{x}^t = v^1(t \bmod L^*) \text{ for } t \in \mathbb{N}_+.$$

It suggests $\boldsymbol{x}^t$ is the first element of $v(t \bmod L^*)$. Therefore, we prove that an optimal solution of (INFTY) is a cyclic policy. The average revenue generated by the cyclic policy is identical to the mean weight of the maximum mean cycle C^*, and the length of the cyclic policy is L^*. If we know L^*, we can compute the cyclic assortment planning via (CYCLE-CONIC). □

EC.4.2. Proof of Theorem 4

THEOREM 4 *Assume that $M < N$. An $(M+1)$-cyclic policy is optimal for problem* (INFTY) *if the cross-period constraint is non-overlapping.*

Proof. We prove this result by showing any L-cyclic policy with length $L > M+1$ will be dominated by the $(M+1)$-cyclic policy in terms of the expected revenue. Recall that we do not consider cyclic polices with length $L < M+1$ because they contain empty sets.

Given an L-cyclic policy satisfying the non-overlapping condition, $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^L$, the purchase probability of product i in period t is

$$\pi_i^t(\boldsymbol{x}^t, \boldsymbol{x}^{t-1}, \ldots, \boldsymbol{x}^{t-M}) = \frac{x_i^t \exp(\beta_i^0)}{1 + \sum_{i\in[N]} x_i^t \exp(\beta_i^0)},$$

because $x_i^t + x_i^{t-1} + \cdots + x_i^{t-M} \le 1$ for $i \in [N]$ and $t \in [L]$. Hence, we can simplify the average revenue of such a cyclic policy as follows

$$\mathrm{Rev}_L = \frac{1}{L}\sum_{t\in[L]} R(\boldsymbol{x}^t) = \frac{1}{L}\sum_{t\in[L]}\sum_{i\in[N]} r_i x_i^t \exp(\beta_i^0)/(1 + \sum_{i\in[N]} x_i^t \exp(\beta_i^0)).$$

For any L-cyclic policy with length $L > M+1$, we show its average revenue is a convex combination of cyclic policies with length $M+1$. We rewrite Rev_L as

$$\begin{aligned}
\mathrm{Rev}_L &= \frac{(M+1)R(\boldsymbol{x}^1) + \cdots + (M+1)R(\boldsymbol{x}^L)}{(M+1)L} \\
&= \frac{\big(R(\boldsymbol{x}^1) + R(\boldsymbol{x}^2) + \cdots + R(\boldsymbol{x}^{M+1})\big) + \cdots + \big(R(\boldsymbol{x}^L) + R(\boldsymbol{x}^1) + \cdots + R(\boldsymbol{x}^M)\big)}{(M+1)L} \\
&= \frac{1}{L}\frac{R(\boldsymbol{x}^1) + R(\boldsymbol{x}^2) + \cdots + R(\boldsymbol{x}^{M+1})}{M+1} + \cdots + \frac{1}{L}\frac{R(\boldsymbol{x}^L) + R(\boldsymbol{x}^1) + \cdots + R(\boldsymbol{x}^M)}{M+1} \\
&\le \mathrm{Rev}^*_{M+1}.
\end{aligned}$$

The first equation holds by multiplying both the denominator and numerator by $M+1$. The second equation holds by re-arranging orders of $R(\boldsymbol{x}^1), \ldots, R(\boldsymbol{x}^L)$. The third equation shows that we construct L cyclic policies with length $M+1$ by abstracting $M+1$ adjacent assortments in the cycle $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^L$. Each cyclic policy is a feasible solution of (CYCLE-CONIC) with length $M+1$ since they are non-overlapping. Then, we have the last inequality because Rev^*_{M+1} is generated by the optimal solution of (CYCLE-CONIC) with length $M+1$. Therefore, the $(M+1)$-cyclic policy has the largest revenue under the non-overlapping condition.

EC.4.3. Proof of Theorem 5

THEOREM 5 $Z_{\text{BOUND-FREE}} \le Z_{\text{CYCLE-CONIC}}$ *if the cross-period constraint is non-overlapping.*

Proof. In the following, we will show that $Z_{\text{BOUND-FREE}} \le Z_{\text{MCCORMICK}} \le Z_{\text{CYCLE-CONIC}}$, where $Z_{\text{MCCORMICK}}$ is the optimal objective value of the continuous relaxation of the following integer programming:

$$\begin{aligned}
\max \quad & \frac{1}{M+1} \sum_{t\in[M+1]} \sum_{i\in[N]} r_i u_i \gamma_i^t \\
\text{s.t.} \quad & x_i^t + \sum_{m\in[M]} x_i^{\tau(m|t)} \le 1 && \text{for } i \in [N] && \text{(EC.8a)} \\
& \rho^t + \sum_{i\in[N]} u_i \gamma_i^t = 1 && \text{for } t \in [M+1] && \text{(EC.8b)} \\
& \gamma_i^t \le \rho_U^t x_i^t && \text{for } i \in [N] \text{ and } t \in [M+1] && \text{(EC.8c)} \\
& \gamma_i^t \le \rho^t + \rho_L^t x_i^t - \rho_L^t && \text{for } i \in [N] \text{ and } t \in [M+1] && \text{(EC.8d)} \\
& \gamma_i^t \ge \rho^t + \rho_U^t x_i^t - \rho_U^t && \text{for } i \in [N] \text{ and } t \in [M+1] && \text{(EC.8e)} \\
& \gamma_i^t \ge \rho_L^t x_i^t && \text{for } i \in [N] \text{ and } t \in [M+1] && \text{(EC.8f)} \\
& \boldsymbol{x}^t \in \mathcal{X} \cap \{0,1\}^N && \text{for } t \in [M+1], && \text{(EC.8g)}
\end{aligned}$$

where constraints (EC.8c)-(EC.8f) are exactly (3), obtained by using the McCormick envelopes to linearize $\gamma_i^t = \rho^t x_i^t$.

First, we prove $Z_{\text{BOUND-FREE}} \le Z_{\text{MCCORMICK}}$. Since the bound-free formulation (12) and formulation (EC.8) have identical objectives, it suffices to show that any feasible solution of the continuous relaxation of (12) is also feasible to that of (EC.8). More specifically, we need to show that constraints (EC.8c)-(EC.8f) are implied by (12b)-(12e) of the bound-free formulation. Constraints (12c), $\Gamma_{ij}^t \le \gamma_i^t$ for $i \ne j$ from (12d), and $\Gamma_{ii}^t = \gamma_i^t$ from (12e) imply that

$$x_i^t \le \left(1 + \sum_{j\in N} u_j\right) \gamma_i^t,$$

which is (EC.8f). Constraints (12b) (12c), $\Gamma_{ij}^t \le \gamma_j^t$ for $i \ne j$ from (12d), and $\Gamma_{ii}^t = \gamma_i^t$ from (12e) imply that

$$x_i^t \le \gamma_i^t + \sum_{j\in[N]} u_j \gamma_j^t = \gamma_i^t + 1 - \rho^t,$$

which is (EC.8e). Also, constraint (12c), $\Gamma_{ij}^t \ge 0$ for $i \ne j$ and $\Gamma_{ii}^t = \gamma_i^t$ imply

$$x_i^t \ge \gamma_i^t,$$

which is (EC.8c). Constraint (12b) (12c), $\Gamma_{ij}^t \ge \gamma_i^t + \gamma_j^t - \rho^t$ for $i \ne j$, $\Gamma_{ii}^t = \gamma_i^t$, and $\gamma_i^t \le \rho^t$ imply

$$\begin{aligned}
x_i^t &\ge \gamma_i^t + u_i \gamma_i^t + \sum_{j\ne i} u_j (\gamma_i^t + \gamma_j^t - \rho^t) = (1 + \sum_{j\in[N]} u_j)(\gamma_i^t - \rho^t) + (1 + u_i^t)\rho^t + \sum_{j\ne i} u_j \gamma_j^t \\
&= (1 + \sum_{j\in[N]} u_j)(\gamma_i^t - \rho^t) + u_i^t \rho^t + 1 - u_i^t \gamma_i^t \ge (1 + \sum_{j\in[N]} u_j)(\gamma_i^t - \rho^t) + 1,
\end{aligned}$$

which is (EC.8d).

Next, we prove $Z_{\text{McCormick}} \leq Z_{\text{Cycle-Conic}}$. Note that in this case, (Cycle-Conic) becomes

$$\begin{aligned}
\max \quad & \frac{1}{M+1} \sum_{i \in [N]} \sum_{t \in [M+1]} r_i y_i^t \\
\text{s.t.} \quad & x_i^t + \sum_{m \in [M]} x_i^{\tau(m|t)} \leq 1 && \text{for } i \in [N] \\
& \rho^t + \sum_{i \in [N]} y_i^t = 1 && \text{for } t \in [M+1] \\
& \max\{0,\ \rho_U^t x_i^{\tau(m|t)} + \gamma_i^t - \rho_U^t\} \leq z_{it}^m \leq \min\{\gamma_i^t, \rho_U^t x_i^{\tau(m|t)}\} && \text{for } t \in [M+1], m \in [M] \quad \text{(EC.9a)} \\
& (3),\ (5),\ (10b) && \text{for } t \in [M+1], i \in [N] \\
& \boldsymbol{x}^t \in \mathcal{X} \cap \{0,1\}^N && \text{for } t \in [M+1],
\end{aligned}$$

where (3) and (EC.9a) are McCormick envelopes linearizing $\gamma_i^t = \rho^t x_i^t$ and $z_{im}^t = \gamma_i^t x_i^{\tau(m|t)}$, respectively. (5) and (10b) are convex and concave extensions of y_i^t, respectively. Letting $(\boldsymbol{x}, \boldsymbol{\rho}, \boldsymbol{\gamma})$ be a feasible solution of the continuous relaxation of (EC.8), and define $y_i^t = \gamma_i^t u_i$ and $z_{im}^t = 0$ for all $t \in [M+1]$ and $i \in [N]$. To complete the proof, we only need to show that $(\boldsymbol{x}, \boldsymbol{\rho}, \boldsymbol{\gamma}, \boldsymbol{y}, \boldsymbol{z})$ is feasible to (Cycle-Conic). Clearly, constraints (3), (5) and (10b) are satisfied. Next, we verify constraint (EC.9a) is satisfied. Clearly, $\min\{\gamma_i^t, \rho_U^t x_i^{\tau(m|t)}\} \geq 0 = z_{im}^t$. On the other hand,

$$\begin{aligned}
\rho_U^t x_i^{\tau(m|t)} + \gamma_i^t - \rho_U^t &\leq \rho_U^t x_i^{\tau(m|t)} + x_i^t \rho_U^t - \rho_U^t \\
&= \rho_U^t (x_i^{\tau(m|t)} + x_i^t - 1) \\
&\leq 0 = z_{im}^t,
\end{aligned}$$

where the first inequality holds because of (EC.8c) and the second inequality holds because of (EC.8a). □

EC.4.4. Example of the Unstable Maximum Mean Cycle Length

The following example shows that a slight fluctuation in one parameter could result in a different maximum mean cycle length if there are no non-overlapping constraints.

Example EC.1. Consider a case where a firm sells three products to customers with a memory length of one. There are no cross-period or cross-product constraints. The outside option has utility 0. Table EC.1 provides a detailed parameter setup. The two-cyclic policy has larger revenue than that of the three-cyclic

Table EC.1 Product parameters

Product	Price	Base utility	One-period history-dependent effect
1	6	1.3	-0.5
2	16	-0.6	-0.8
3	17	0.2	-0.9

policy. However, once the base utility of product 3 increases from 0.2 to 0.3, the average revenue of the two-cyclic policy is smaller than that of the three-cyclic policy. □

EC.5. Additional Material to Section 6

EC.5.1. Multilinear Extension Based Formulation

In this subsection, we introduce the multilinear extension based formulation of (HAP). For the nonlinear structure (CHOICE), we first use the multilinear extension to present the attraction value function $\alpha_i(\cdot)$ (O'Donnell 2014), and then recursively apply McCormick envelopes to derive a linear representation. It is well-understood that both the quality and the size of the resulting linear representations depend on the recursive sequence, and finding an optimal recursive sequence amounts to solving a difficult combinatorial optimization problem (Speakman and Lee 2017, Khajavirad 2023). Speakman and Lee (2017) show that the relaxation quality depends on the sequence of recursion, but we do not exploit the best sequence in this paper. Even though the number of introduced variables and constraints increases exponentially with the memory length, this formulation is solvable under a small M.

We represent the attraction value function via a sum of multiple multilinear functions. Given memory length $M \in \mathbb{Z}_+$, $\{S_k \mid S_k \subseteq [M], k \in [2^M]\}$ is the collection of all subsets of $[M]$. For every $S_k, k \in [2^M]$, let χ^{S_k} be its indicator vector, that is, the m^{th} coordinate of χ^{S_k} is 1 if and only if $m \in S_k$. We can rewrite the attraction value function as

$$\alpha_i(x_i^{t-1}, \dots, x_i^{t-M}) = \sum_{k \in [2^M]} \left(\Pi_{m \in S_k} x_i^{t-m}\right) \cdot a_{ik} \quad \text{for } i \in [N],$$

where $a_{ik} = \alpha_i(\chi^{S_k}) - \sum_{j: S_j \subset S_K} a_{ij}$.

Then, introduce θ_{ik}^t to present the multilinear term, that is,

$$\theta_{ik}^t = x_i^t \cdot \left(\Pi_{m \in S_k} x_i^{t-m}\right) = \Pi_{m \in S_k \cup \{0\}} x_i^{t-m} \quad \text{for } i \in [N], t \in [T], k \in [2^M].$$

We recursively use McCormick envelopes to linearize θ_{ik}^t. The idea is to sequentially introduce artificial variables and reduce the number of variables in the multilinear term. First, select any two components $p, q \in S_k \cup \{0\}$. Then, introduce a variable $h_i^t(p,q)$ that corresponds to the bilinear product $x_i^{t-p} x_i^{t-q}$ and the multilinear term changes to

$$\theta_{ik}^t = h_i^t(p,q) \cdot \Pi_{m \in S_k \cup \{0\} \setminus \{p,q\}} x_i^{t-m}.$$

We can use McCormick envelopes to linearize $h_i^t(p,q) = x_i^{t-p} x_i^{t-q}$,

$$\max\{0, x_i^{t-p} + x_i^{t-q} - 1\} \le h_i^t(p,q) \le \min\{x_i^{t-p}, x_i^{t-q}\}.$$

This procedure can be recursively applied to the remaining parts until θ_{ik}^t is completely linearized. We use a notation $\mathcal{H}(i,t,k)$ to present the corresponding McCormick envelopes.

Note that $\theta^t_{i,k} \in \{0,1\}$, we also apply the McCormick envelopes to linearize $\rho^t\theta^t_{ik}$ denoted by w^t_{ik}. That is,

$$w^t_{ik} \le \min\{\rho^t_L\theta^t_{ik} + \rho^t - \rho^t_L,\ \rho^t_U\theta^t_{ik}\} \tag{EC.10a}$$

$$w^t_{ik} \ge \max\{\rho^t_L\theta^t_{ik},\ \rho^t_U\theta^t_{ik} + \rho^t - \rho^t_U\}. \tag{EC.10b}$$

Finally, we obtain a mixed-integer linear formulation of (HAP) as follows.

$$\begin{aligned}
\max \quad & \frac{1}{T}\sum_{t\in[T]}\sum_{i\in[N]} r_i y^t_i & & \\
\text{s.t.} \quad & \boldsymbol{x}^t \in \{0,1\}^N \cap \mathcal{X} \text{ and } \boldsymbol{x} \in \mathcal{P} & & \text{for } t\in[T] \\
& \rho^t + \sum_{i\in[N]} y^t_i = 1 & & \text{for } t\in[T] \\
& y^t_i = \sum_{k\in[2^M]} w^t_{ik} a_{ik} & & \text{for } t\in[T] \text{ and } i\in[N] \\
& \theta^t_{ik} \in \mathcal{H}(i,t,k) & & \text{for } t\in[T], i\in[N] \text{ and } k\in[2^M] \\
& \text{(EC.10)} & & \text{for } t\in[T], i\in[N] \text{ and } k\in[2^M].
\end{aligned} \tag{MULTILINEAR}$$

EC.5.2. A Projected Cutting-Plane Implementation for Bound-Free Formulation (12)

The bound-free formulation (12) has $\mathcal{O}(N^2)$ variables due to the presence of $\boldsymbol{\Gamma}$ variables. We design a cutting-plane procedure to implement (12) for computation efficiency. Recall that (12) is reformulation of ((M+1)-CYCLIC). In our implementation, we first reformulate ((M+1)-CYCLIC) as a base model. Next, we use a projected separation oracle to cut off infeasible solutions for K rounds.

We linearize ((M+1)-CYCLIC) to a base model as follows. First, following the idea in Section 3.1, we introduce variable ρ^t to denote $1/(1+\sum_{j\in[N]} u_j x^t_j)$ with constraint $\rho^t + \sum_{j\in[N]} u_j x^t_j \rho^t = 1$. Next, let γ^t_j denote the bilinear product $x^t_j\rho^t$ and apply McCormick envelopes to linearize the bilinear term. We also notice that, for $t\in[M+1]$,

$$\rho^t(1+\sum_{j\in[N]} x^t_j u_j) \ge 1,$$

which is a valid convex constraint and is representable via the second-order cone. Let $w^t = 1 + \sum_{j\in[N]} x^t_j u_j$ for $t \in [M+1]$ and add the above constraint into our formulation. Finally, we obtain a mixed-integer second-order conic formulation of ((M+1)-CYCLIC) as follows.

$$\begin{aligned}
\max \quad & \frac{1}{M+1}\sum_{t\in[M+1]}\sum_{i\in[N]} r_i u_i \gamma_i^t \\
\text{s.t.} \quad & x_i^t + \sum_{m\in[M]} x_i^{\tau(m|t)} \le 1 && \text{for } i\in[N] \\
& \rho^t + \sum_{i\in[N]} u_i\gamma_i^t = 1 && \text{for } t\in[M+1] \\
& \gamma_i^t \le \min\{\rho_L^t x_i^t + \rho^t - \rho_L^t,\ \rho_U^t x_i^t\} && \text{for } i\in[N], t\in[M+1] \\
& \gamma_i^t \ge \max\{\rho_L^t x_i^t,\ \rho_U^t x_i^t + \rho^t - \rho_U^t\} && \text{for } i\in[N], t\in[M+1] \\
& \rho^t + w^t \ge ||\rho^t - w^t, 2|| && \text{for } t\in[M+1] \\
& w^t = 1 + \sum_{j\in[N]} x_j^t u_j && \text{for } t\in[M+1] \\
& \boldsymbol{x}^t \in \{0,1\}^N && \text{for } t\in[M+1],
\end{aligned} \tag{BASE}$$

where ρ_L^t and ρ_U^t are lower and upper bound of ρ^t for $t\in[M+1]$, respectively. Here, although we need the bound of ρ^t when implementing (12), we do not require it to be very tight, and we use it only for getting an equivalent formulation of ((M+1)-CYCLIC).

We tighten the base formulation through a projected separation oracle (Algorithm 2). Given a solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\rho}})$, the separation oracle checks whether the projection of (12c) in the space of the base formulation is violated for each $i\in[N]$ and $t\in[M+1]$. More specifically, we replace Γ_{ij}^t in (12c) with the bounds in (12d) and (12e), then we have

$$\gamma_i^t(1+u_i) + \sum_{j\in[N], j\neq i} u_j \min\{0, \gamma_i^t + \gamma_j^t - \rho^t\} \le x_i^t \le \gamma_i^t + \sum_{j\in[N]} u_j \min\{\gamma_i^t, \gamma_j^t\}. \tag{EC.11}$$

If the current solution violates (EC.11), we add the corresponding constraint to the base model. We detail each step in Algorithm 2.

Running the separation oracle once is a round of cut generation. We add new cuts for K rounds and the final formulation as BF-K, where BF denotes “bound-free”. In each round $k\in[K]$, the cutting-plane algorithm solves the continuous relaxation of current formulation BF-$(k-1)$ and obtain an optimal solution $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\rho}})$. Then, the algorithm calls the separation oracle (Algorithm 2) to generate new cuts. The algorithm finally solves BF-K with binary constraints and obtains an optimal integer solution. We present details of this implementation process in Algorithm 3.

EC.5.3. A Cutting-Plane Implementation for Large Memory Length

One of the main difficulties of solving (CONIC) is that for each $i\in[N]$ and $t\in[T]$, the number of linear inequalities used to describe the concave envelope of the attraction value function in (7) is $M!$, which grows factorially as M becomes large. In the following, we will implement constraints in (7) in the so-called lazy

Algorithm 2 Projection-based separation oracle for (BASE)

Input: $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\rho}})$ and base utility $\boldsymbol{u}$
Cuts $\leftarrow \emptyset$, $A \leftarrow \emptyset$, $B \leftarrow \emptyset$, and $C \leftarrow \emptyset$
for $i \in [N]$ and $t \in [M+1]$ **do**
  for $j \in [N]$ **do**
    if $\hat{\gamma}_i^t + \hat{\gamma}_j^t \geq \hat{\rho}_t$ and $i \neq j$ **then**
      $A \leftarrow A \cup \{j\}$
    end if
    if $\hat{\gamma}_j^t \geq \hat{\gamma}_i^t$ **then**
      $B \leftarrow B \cup \{j\}$
    else
      $C \leftarrow C \cup \{j\}$
    end if
  end for
  lower-bound $\leftarrow (1+u_i) \cdot \hat{\gamma}_i^t + \sum_{j \in A} u_j (\hat{\gamma}_i^t + \hat{\gamma}_j^t - \hat{\rho}_t)$
  upper-bound $\leftarrow \hat{\gamma}_i^t + \sum_{j \in B} u_j \hat{\gamma}_i^t + \sum_{j \in C} u_j \hat{\gamma}_j^t$
  if $\hat{x}_i^t \leq$ lower-bound **then**
    Cuts $\leftarrow x_i^t \geq (1+u_i)\gamma_i^t + \sum_{j \in A} u_j (\gamma_i^t + \gamma_j^t - \rho_t)$
  else if $\hat{x}_i^t \geq$ upper-bound **then**
    Cuts $\leftarrow x_i^t \leq \gamma_i^t + \sum_{j \in B} u_j \gamma_i^t + \sum_{j \in C} u_j \gamma_j^t$
  end if
end for
Output: Cuts

Algorithm 3 A projected-cutting-plane implementation of bound-free formulation (12)

Input: Formulation (BASE) (denoted as BF-0) and a positive integer K
$k \leftarrow 1$
while $k \leq K$ **do**
  $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\rho}}) \leftarrow$ an optimal solution to the continuous relaxation of BF-$(k-1)$
  Cuts $\leftarrow$ Call the separation oracle, Algorithm 2, to separate$(\hat{\boldsymbol{x}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{\rho}})$
  BF-k $\leftarrow$ Add Cuts into BF-$(k-1)$formulation
  $k \leftarrow k+1$
end while
Solve BF-K with binary constraints of $\boldsymbol{x}$
Output: An optimal solution of BF-K, $(\boldsymbol{x}^*, \boldsymbol{\gamma}^*, \boldsymbol{\rho}^*)$

fashion. More specifically, we use the `Callback` routine of `Gurobi` to implement (7) as lazy constraints. Constraints in (7) are removed from (CONIC) and placed in a lazy constraint pool. Then, `Gurobi` solves the relaxed problem and checks whether inequalities in the pool are violated at each integer solution generated in the branch-and-bound tree. If a violated inequality is found, it is added to the node, the integer solution is cut off, and the node is resolved. We elaborate on our implementation as follows.

Since `Gurobi` does not support the exponential function, we linearize the exponential conic constraint (5) using the subgradient inequalities of the continuous relaxation of $\alpha_i(\cdot)$ and treat the subgradient inequalities as a lazy constraint as well. Note that the continuous relaxation $\tilde{\alpha}_i(\cdot)$ is a closed convex function over $[0,1]^M$. Thus, by Rockafellar (1970), for $(x_i^{t-1},\dots,x_i^{t-M}) \in [0,1]^M$

$$\tilde{\alpha}_i(x_i^{t-1},\dots,x_i^{t-M}) = \max_{\boldsymbol{w}} \left\{ \tilde{\alpha}_i(\boldsymbol{w}) + \sum_{m\in[M]} (x_i^{t-m} - w^m)\beta_i^m \tilde{\alpha}_i(\boldsymbol{w}) \;\middle|\; \boldsymbol{w} \in [0,1]^M \right\}.$$

where $\beta_i^m \tilde{\alpha}_i(\boldsymbol{w})$ is the m^{th} partial derivate of $\nabla \tilde{\alpha}_i(\boldsymbol{w})$. Therefore, using the variable γ_i^t to scale the subgradient inequalities, the exponential conic constraint (5) is equivalent to

$$y_i^t \geq \gamma_i^t \alpha_i(\boldsymbol{w}) + \sum_{m\in[M]} (z_{im}^t - \gamma_i^t w^m)\beta_i^m \alpha_i(\boldsymbol{w}) \qquad \text{for } \boldsymbol{w} \in [0,1]^M. \tag{EC.12}$$

To implement (7) and (EC.12) as lazy constraints in `Gurobi`, we use Algorithm 4 to solve the separation problem. Given a point (solution) $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{z}})$ in the space of these two constraints, for each product $i \in [N]$ and period $t \in [T]$, we check whether constraints in (7) and (EC.12) is violated. For constraint (7), we obtain a permutation $\hat{\sigma}$ by sorting $\{g_{im}^t\}_{m\in[M]}$ in a non-increasing order, that is, $g_{i\hat{\sigma}(1)}^t \geq \cdots \geq g_{i\hat{\sigma}(M)}^t$, where $g_{im}^t = \hat{z}_{im}^t$ if $m \notin I_i$ and $g_{im}^t = \hat{\gamma}_i^t - \hat{z}_{im}^t$ if $m \in I_i$. The constraint in (7) corresponding to the permutation $\hat{\sigma}$ is

$$y_i^t \leq \alpha_i(\boldsymbol{h}_{i0}^{\hat{\sigma}})(\gamma_i^t - \tilde{z}_{i\hat{\sigma}(1)}^t) + \sum_{k\in[M]} \alpha_i(\boldsymbol{h}_{ik}^{\hat{\sigma}})(\tilde{z}_{i\hat{\sigma}(k)}^t - \tilde{z}_{i\hat{\sigma}(k+1)}^t). \tag{EC.13}$$

The given point $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{z}})$ satisfies (7) if and only if it satisfies the constraint (EC.13). For the convex part, we directly check the subgradient constraint obtained at historical assortments $(\hat{w}^1,\dots,\hat{w}^M) = (\hat{z}_{i1}^t/\hat{\gamma}_i^t,\dots,\hat{z}_{iM}^t/\hat{\gamma}_i^t)$:

$$y_i^t \geq \gamma_i^t \alpha_i(\hat{w}^1,\dots,\hat{w}^M) + \sum_{m\in[M]} (z_{im}^t - \gamma_i^t \hat{w}^m)\beta_i^m \alpha_i(\hat{w}^1,\dots,\hat{w}^M). \tag{EC.14}$$

The given point $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{z}})$ satisfies (EC.12) if and only if it satisfies the constraint (EC.14). We summarizes the above procedure in Algorithm 4

The idea of separating constraint (7) is as follows. Since constraint (7) is the scaled Lovász extension of the attraction value function, the separation problem reduces to identifying the correct linear inequality from its piecewise-linear representation. Each linear piece corresponds to a specific ordering (permutation) of the coordinates, that is, the historical assortments $(x_i^{t-1},\dots,x_i^{t-M})$. Given a candidate solution $\hat{\boldsymbol{x}}$, the permutation $\hat{\sigma}$ obtained by sorting the historical assortments in a non-increasing order identifies which simplex contains the current point and determines the linear interpolation that yields the Lovász extension. For example, when the memory length is two, in Figure 2a, nodes in the blue shaded triangle satisfy $x_i^{t-1} \geq x_i^{t-2}$, which corresponds to permutation $(1,2)$. This permutation maps to the nested vertices $(0,0)-(1,0)-(1,1)$ that define the Lovász extension for these nodes. In the mixed case shown in Figure 2b, we sort the switched historical assortments, i.e., $1-x_i^{t-1} \geq x_i^{t-2}$, again obtaining permutation $(1,2)$. Since the linear scaling (by γ_i^t) preserves the relative ordering, we can directly sort g_{im}^t, as done in our algorithm. We provide a formal proof of the separation procedure of both constraint (7) and (EC.12) in the following Lemma EC.4.

LEMMA EC.4. *Let $\mathcal{Y} = \big\{(\boldsymbol{y}, \boldsymbol{\gamma}, \boldsymbol{z}) \mid (7), (\text{EC.12})\big\}$. Algorithm 4 solves the separation problem of set $\mathcal{Y}$ in polynomial time.*

Proof of Lemma EC.4: We need to show that if current solution $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{z}}) \notin \mathcal{Y}$, then the returned cuts can separate current solution and $\mathcal{Y}$. First, it is easy to check that the returned cuts are valid inequalities for $\mathcal{Y}$. That is, each point $(\boldsymbol{y}, \boldsymbol{\gamma}, \boldsymbol{z}) \in \mathcal{Y}$ satisfies (EC.13) and (EC.14).

Next, we discuss two cases when the returned cuts could cut off the current solution. We argue that it is sufficient to consider positive $\hat{\gamma}_i^t$ because the integer solution must be $\hat{x}_i^t = \hat{\gamma}_i^t = z_{i1}^t = \cdots = z_{iM}^t = \hat{y}_i^t = 0$ if $\hat{\gamma}_i^t = 0$, which is feasible. If $(\boldsymbol{y}, \boldsymbol{\gamma}, \boldsymbol{z})$ violates at least one constraint (7). That is,

$$\begin{aligned}
\hat{y}_i^t &> \hat{\gamma}_i^t \alpha_i(\hat{z}_{i1}^t/\hat{\gamma}_i^t, \ldots, \hat{z}_{iM}/\hat{\gamma}_i^t) \\
&= \min_{\sigma} \big\{ \alpha_i(\boldsymbol{h}_{i0}^{\sigma})(\hat{\gamma}_i^t - g_{i\sigma(1)}^t) + \sum_{k \in [M]} \alpha_i(\boldsymbol{h}_{ik}^{\sigma})(g_{i\sigma(k)}^t - g_{i\sigma(k+1)}^t) \big\} \\
&= \alpha_i(\boldsymbol{h}_{i0}^{\hat{\sigma}})(\hat{\gamma}_i^t - g_{i\hat{\sigma}(1)}^t) + \sum_{k \in [M]} \alpha_i(\boldsymbol{h}_{ik}^{\hat{\sigma}})(g_{i\hat{\sigma}(k)}^t - g_{i\hat{\sigma}(k+1)}^t),
\end{aligned}$$

where g_{im}^t is switched $\hat{z}_{im}^t$ defined in Algorithm 4. The fist and second equalities hold by the definition of the Lovász function of $\alpha_i(\cdot)$. Clearly, such a solution can be cut off by (EC.13). If $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{z}})$ violates at least one constraint of (EC.12). That is,

$$\hat{y}_i^t < \hat{\gamma}_i^t \alpha_i(\hat{z}_{i1}^t/\hat{\gamma}_i^t, \ldots, \hat{z}_{iM}^t/\hat{\gamma}_i^t) = \hat{\gamma}_i^t \alpha_i(\hat{w}^1, \ldots, \hat{w}^M) + \sum_{m \in [M]} (\hat{z}_{im}^t - \hat{\gamma}_i^t \hat{w}^m) \beta_i^m \alpha_i(\hat{w}^1, \ldots, \hat{w}^M),$$

where the equality hold by the definition $\hat{w}^m = \hat{z}_{im}^t / \hat{\gamma}_i^t$. Such a solution can be cut off by (EC.14). $\square$

The last element in our implementation is to select a starting formulation before calling `Gurobi`. We solve a collection of relaxed models to obtain continuous solutions and call the separation oracle to add new cuts. The relaxed models contain a subset of constraints (7) and (EC.12), defined as follows:

$$\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{\rho},\boldsymbol{\gamma},\boldsymbol{z}} \quad & \frac{1}{T}\sum_{t\in[T]}\sum_{i\in[N]} r_i y_i^t \\
\text{s.t.} \quad & \boldsymbol{x}^t \in [0,1]^N \cap \mathcal{X} \text{ and } \boldsymbol{x}\in\mathcal{P} && \text{for } t\in[T] \\
& \rho^t + \sum_{i\in[N]} y_i^t = 1 && \text{for } t\in[T] \\
& (3),\ (4) && \text{for } t\in[T] \text{ and } i\in[N] \\
& (7) && \text{for } \sigma\in\Omega_{it}^k, t\in[T] \text{ and } i\in[N] \\
& (\text{EC.12}) && \text{for } \tilde{\boldsymbol{w}}\in W_{it}^k, t\in[T] \text{ and } i\in[N],
\end{aligned} \tag{$\textsc{Rlx}_k$}$$

where $\Omega_{it}^k \subset \Omega$ (resp. $W_{it}^k \subset [0,1]^M$) is a subset of permutations (resp. nodes). Given a continuous solution, we call the separation oracle (Algorithm 4) to add new cuts and update $\textsc{Rlx}_k$ to $\textsc{Rlx}_{(k+1)}$. We add K rounds of cuts and set the final $\textsc{Rlx}_K$ with binary constraints as a starting model for `Callback`. We present details of the implementation process in Algorithm 5.

Algorithm 4 Separation oracle for (7) and (EC.12)

Input: $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{z}})$
Cuts$\leftarrow \emptyset$
for $i\in[N]$ and $t\in[T]$ **do**
  Generate $\hat{\sigma}$ by sorting $\{g_{i\hat{\sigma}(m)}^t\}_{m\in[M]}$ such that $g_{i\hat{\sigma}(1)}^t \geq \cdots \geq g_{i\hat{\sigma}(M)}^t$, where $g_{im}^t = \hat{z}_{im}^t$ if $m\notin I_i$ and $g_{im}^t = \hat{\gamma}_i^t - \hat{z}_{im}^t$ if $m\in I_i$.
  if (EC.13) is violated **then**
    Cuts$\leftarrow$ (EC.13)
  end if
  Find the historical assortments $(\hat{w}^1,\ldots,\hat{w}^M) = (\hat{z}_{i1}^t/\hat{\gamma}_i^t,\ldots,\hat{z}_{iM}^t/\hat{\gamma}_i^t)$
  if (EC.14) is violated **then**
    Cuts$\leftarrow$ (EC.14)
  end if
end for
Output: Cuts

EC.6. Supplementary Numerical Studies

EC.6.1. Performance under Cross-product and Cross-period Constraints

In this numerical study, we add cross-product and cross-period constraints to understand how such constraints impact the computational performance of our formulations. We consider the cardinality constraint

Algorithm 5 A cutting-plane algorithm for (CONIC) with large memory length

Input: Parameters $\boldsymbol{r}, \{\beta_i^0\}_{i\in[N]}, \boldsymbol{\beta}, M, N, T, \{\Omega_{it}^1, W_{it}^1\}_{i\in[N],t\in[T]}$, and K

$k \leftarrow 1$

while $k \leq K$ **do**

  $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{z}})$ Solve (RLX_k)

  Cuts $\leftarrow$ Call the separation oracle, Algorithm 4, to separate $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\gamma}}, \hat{\boldsymbol{z}})$

  $\text{RLX}_{k+1} \leftarrow$ Add Cuts into RLX_k

  $k \leftarrow k+1$

end while

Add binary constraints to the final RLX_K and solve it through the `Callback` routine of `Gurobi` to call the separation oracle Algorithm 4

Output: An optimal integer solution $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{\rho}^*, \boldsymbol{\gamma}^*, \boldsymbol{z}^*)$

for each period:

$$\sum_{i\in[N]} x_i^t \leq C \quad \text{for } t \in [T]. \tag{EC.15}$$

That is, the number of offered products in each period is no more than C, a positive integer. For the cross-period constraint, we consider that the number of times provided for a product $i \in [N]$ shall not exceed K. That is,

$$\sum_{t\in[T]} x_i^t \leq K \quad \text{for } i \in [N]. \tag{EC.16}$$

We focus on the satiation effects and generate parameters as follows. The revenue and base utility of a product is randomly generated from uniform distributions with ranges of $[1, 10]$ and $[-1, 1]$, respectively. The history-dependent effect β_i^1 is uniformly sampled from $[-1, 0]$ for each $i \in [N]$. We fix $N = 30, T = 5$ and $M = 1$. We generate 5 synthetic instances for each configuration of $C \in \{5, 10, 15, 30\}$ and $K \in \{2, 3, 4, 5\}$. We use the MILP model, Env, to solve each instance within 3600 seconds and with an optimality gap 0.5%.

Table EC.2 summarizes the computation performance of Env under constraints of (EC.15) and (EC.16). The first two columns are the limitation of cardinality size and offering times, respectively. The third column #sol indicates the number of instances solved to optimality within 3600 seconds. T_{opt} is the average computation time of solved instances and G_{end} is the average end gap of unsolved instances. The last column G_{root} is the average root gap of all instances. Table EC.2 shows that our formulation is more sensitive to the cross-period offering constraint (EC.16) than the cross-product cardinality constraint (EC.15). For instance, under the same offering times K, the average computation time, end gap, and root gap are stable under different cardinality sizes C. However, under the same C, if we increase offering times K from 2

Table EC.2 Computation time, end gap, and root gap of Env formulation under weak satiation effects with cardinality constraint (EC.15) and offering constraint (EC.16) ($N = 30, T = 5, M = 1$)

C	K	#sol	$T_{\text{opt}(s)}$	$G_{\text{end}(\%)}$	$G_{\text{root}(\%)}$	C	K	#sol	$T_{\text{opt}(s)}$	$G_{\text{end}(\%)}$	$G_{\text{root}(\%)}$
5	2	1	801.20	2.15	13.58	15	2	1	796.22	1.93	13.23
5	3	4	509.65	1.73	7.49	15	3	4	385.27	1.46	7.03
5	4	4	369.20	1.28	5.81	15	4	4	207.41	0.86	4.86
5	5	4	139.18	1.18	4.41	15	5	5	86.74	0	3.26
10	2	1	960.74	1.96	13.22	30	2	1	871.66	1.86	13.22
10	3	4	339.98	1.42	7.03	30	3	4	439.71	1.44	7.03
10	4	4	187.44	0.71	4.86	30	4	4	214.48	0.93	4.86
10	5	5	82.12	0	3.27	30	5	5	89.89	0	3.26

to 4, the average computation time, end gap, and root gap shrink by more than one-half. It is because our formulation does not characterize the problem structure in the space of cross-period choice probabilities, (y_i^t, y_i^{t+1}). Although not all instances are solved to optimality, the end gap is no more than 2.5%. It indicates the robustness and efficiency of our formulation.

EC.6.2. Performance of Instances with Large Memory Length

This numerical study demonstrates that our (CONIC) formulation combined with the cutting-plane algorithm can solve instances with moderately large memory length.

We fix the number of products $N = 20$ and planning horizon $T = M + 1$ for $M \in \{4, 5, 6\}$. We consider both addiction and satiation effects. Specifically, a product has an addiction (resp. satiation) effect with probability θ (resp. $1 - \theta$), where $\theta \in \{0, 0.1, 0.2\}$. For products with addiction (resp. satiation) effects, its $\{\beta_i^m\}_{m\in[M]}$ are uniformly sampled from $[0, 1]$ (resp. $[-1, 0]$). For each configuration of M, T, and θ, we randomly generate 10 instances. The revenue and the base utility of a product are from uniform distributions with ranges of $[1, 10]$ and $[-1, 1]$, respectively. To initialize the cutting-plane algorithm, we set Ω_{it}^1 by randomly generating 2,10, and 20 different permutations over $[M]$ for memory lengths of 4,5, and 6, respectively. We initialize W_{it}^1 by adding $\boldsymbol{w} \in \{0, 1\}^M$ satisfying $\|\boldsymbol{w}\|_2^2 \in \{0, 1, 2, M\}$. When computing the starting model for `Callback`, we do not fix K rounds to add cuts. Instead, we call the separation oracle until the reduction of the objective value of (RLX_k) is less than a small constant ϵ. We set $\epsilon = 10^{-8}$ in the numerical study.

Table EC.3 summarizes the computation results, where we set a time limit of 7200 seconds and an optimality gap tolerance of 0.5%. The first three columns show the memory length, planning horizon, and the proportion of products with positive effects. The last three columns present the number of solved instances, the average time of solved instances (including set-up time of the starting model and `Callback` time), and the average end gap of unsolved instances. Table EC.3 shows that the cutting-plane algorithm can solve most instances to optimality within two hours, even when $M = 6$. For instance, when $M = 6$ and $\theta = 0.2$, 7 instances are solved to optimality using only 18.16 seconds on average. While some instances cannot achieve optimality within two hours, their end gaps are small and no more than 4%.

Table EC.3 **Average computation time of solved instanced and average end gap of unsolved instances**

M	T	θ	#sol	Time (s)	$G_{\text{end}(\%)}$
4	5	0	10	317.13	0
4	5	0.1	10	2.94	0
4	5	0.2	10	17.00	0
5	6	0	9	273.89	1.6
5	6	0.1	9	1632.32	3.19
5	6	0.2	10	170.72	0
6	7	0	6	3416.57	2.96
6	7	0.1	8	963.53	3.8
6	7	0.2	7	18.16	2.75

EC.6.3. Performance of $(M+1)$-cyclic Policy

In this numerical study, we evaluate the $(M+1)$-cyclic policy on synthetic data and compare its performance with $\mathsf{Sequential-RO}$ and $\mathsf{Sequential-LOSPO}$. We fix the number of products $N=30$ and vary the memory length $M\in\{1,2\}$ and planning horizon $T\in\{10,30,50\}$. We generate product revenue, base utility, and history-dependent effects in the same way as the numerical study in Section 6.1 and only consider strong satiation effects. For each combination of M and T, we generate five instances.

Table EC.4 shows the average revenue gap of the $(M+1)$-cyclic policy and two heuristics against the best feasible solution obtained by solving the envelope-based formulation, ENV, in 7200 seconds. In general, the $(M+1)$-cyclic policy has a very small revenue loss, which is no more than 1% in most cases and about only 2% of revenue losses of the two heuristics. The revenue gap can even be negative, indicating that the cyclic policy can outperform the best feasible solution obtained within the time limit. Moreover, the revenue gap shrinks when T increases, which matches the asymptotical optimality of the $(M+1)$-cyclic policy.

Table EC.4 **Revenue gap ($\%$) of $(M+1)$-cyclic policy and heuristics ($N=30$)**

M	T	$(M+1)$-cyclic	$\mathsf{Sequential-RO}$	$\mathsf{Sequential-LOSPO}$
1	10	0.5	12.79	11.42
1	30	0.01	14.51	13.16
1	50	-0.08	14.86	13.51
2	10	1.03	27.63	26.12
2	30	-0.48	35.94	34.79
2	50	-0.68	37.74	36.67